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The limitations on applying classical thin plate theory to thin annular plates clamped on the inner boundary

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The experimentally measured resonance frequencies of a thin annular plate with a small ratio of inner to outer radii and clamped on the inner boundary are compared to the predictions of classical thin-plate (CTP) theory and a finite-element (FE) model. The results indicate that, contrary to the conclusions presented in a number of publications, CTP theory does not accurately predict the frequencies of a relatively small number of resonant modes at lower frequencies. It is shown that these inaccuracies are attributable to shear deformations, which are thought to be negligible in thin plates and are neglected in CTP theory. Of particular interest is the failure of CTP theory to accurately predict the resonance frequency of the lowest vibrational mode, which was shifted approximately 30% by shear motion at the inner boundary. *Copyright 2012 Author(s). This article is distributed under a Creative Commons Attribution 3.0 Unported License*. [http://dx.doi.org/10.1063/1.4757928]

I. INTRODUCTION

The vibrations of circular flat plates have been of intrinsic interest for over a century.[1–5] The interest in such a seemingly simple physical system is due to the widespread application of this geometry, as well as the inherent ability to study it in detail both theoretically and experimentally. Although within the past few years it has become common to analyze vibrating systems using commercially available finite element modeling packages, due to the inherent simplicity of the system it is still common to use Kirchhoff thin-plate theory, also known as the classical thin-plate (CTP) theory, to analyze the motion of vibrating plates. CTP theory is especially useful when an understanding of the physics of plate motion is important and merely predicting an accurate result using a finite element program is not sufficient.

CTP theory does not include shear deformations and is only applicable for plates having a ratio of thickness to diameter of less than approximately 0.05. When plates are this thin, shear motion is believed to be negligible and therefore can be neglected in the analysis. When the thickness exceeds this limit a theory that takes transverse shear strain into account must be used to analyze the plate.[6] It is also widely accepted that CTP theory is adequate only for predicting the lowest modes of vibration even for very thin plates.[7–9] Specifically, earlier studies indicate that CTP theory "underestimates deflections and overestimates frequencies" of higher modes.[10]

It is important to note that CTP theory is commonly used to analyze the behavior of thin plates with any boundary condition: free, simply supported or clamped.[11] In all of these cases, if the plate is thin, it is commonly believed that the shear motion of the plate need not be considered in the analysis because it is negligible. In the work reported here we consider a thin annular plate that is free on the outer edge and clamped on the inner edge. The clamped boundary is assumed to completely restrict the out-of-plane motion, but because the plate is thin and the in-plane shear motion is assumed to be negligible, in-plane motion need not be restricted in order to apply CTP theory to the analysis.

In what follows, we present the results of experiments and theoretical analyses of a thin annular plate with a small ratio of inner to outer radii in which the majority of the first 22 resonance

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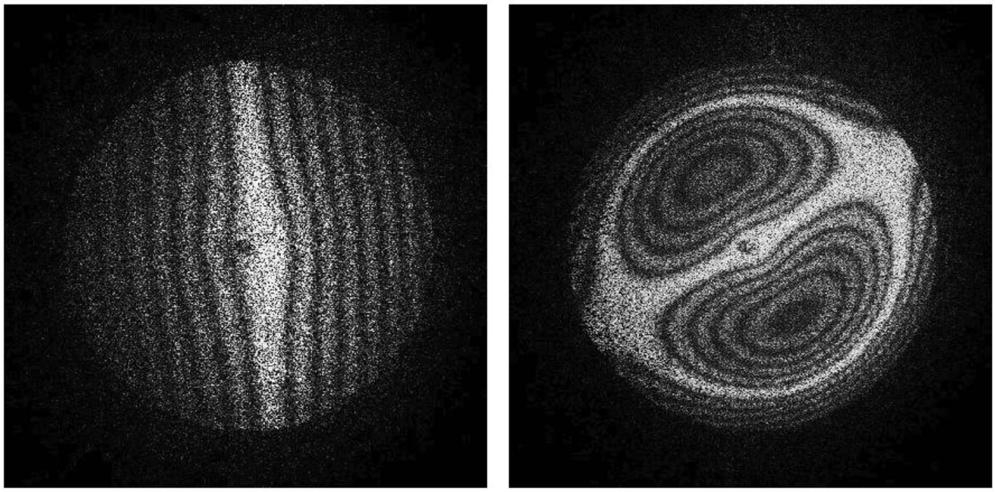

(a) (1,0) mode (b) (1,1) mode

FIG. 1. Interferograms showing two of the normal mode deflection shapes of a thin, annular plate. Nodal lines are white; black and gray lines are contours of equal displacement. (a) (1, 0) mode oscillating at 74 Hz, (b) (1, 1) mode oscillating at 771 Hz.

frequencies, which range as high as 55 times the value of the lowest resonance frequency, are well predicted by CTP theory, while some of the lowest order modes of vibration are not; these results appear to be in conflict with several earlier studies. We demonstrate that the differences between the predictions of CTP theory and the experimental results for some of the lower modes are due to the effects of transverse shear motion at the inner boundary.Therefore, we conclude that transverse shear motion can be important even in very thin plates, contrary to what is commonly stated in the literature.

II. EXPERIMENTS

The plate under consideration is a 2.0 ± 0.1 mm thick annular glass plate with an outer diameter of 256.6 ± 0.2 mm and an inner diameter of 8.6 ± 0.2 mm. The plate was secured to stainless steel post by a bolt in such a way that the inner radius was clamped between two washers with a diameter of 12.5 ± 0.1 mm. The diameter of the washers was smaller than diameter of the post and the post was secured to a two-inch diameter steel post which was mounted on an optical table that was actively isolated from ambient vibrations.

The resonance frequencies of the plate and the corresponding normal mode patterns were determined using time-averaged electronic speckle pattern interferometry. The light source was a frequency-doubled Nd:YVO_4 laser with a wavelength of 532 nm. A complete description of this interferometric arrangement can be found elsewhere.[12] The glass plate was excited acoustically by a speaker placed approximately 0.5 m away, which was driven by a function generator that produced a sinusoidal signal with a precision of ± 0.05 Hz. Interferograms of two of the normal modes of interest are shown in Fig. 1. In interferograms such as those shown in Fig. 1, areas of white indicate nodal lines while alternating lines of black and gray represent contours of equal displacement. Note that the interferograms indicate that the plate had a high degree of symmetry, and indeed the degeneracy between orthogonal modes was not broken by more than a 0.1% in most cases.

The modal shape of the plate was recorded using electronic speckle pattern interferometry when it was excited at all of the resonance frequencies discussed below. In addition to allowing the unambiguous identification of the modal shapes, the lack of any motion of the internal boundary was confirmed within the limits of detection of the interferometer (~100 nm). The lack of any indication of out-of-plane motion of the support demonstrated that the out-of-plane vibrations were effectively damped and that the plate could be considered an annular plate clamped on the inner boundary,

TABLE I. Measured frequencies at which the normal modes of a glass flat plate occur beside the frequencies predicted by CTP theory.

Frequencies of Normal Modes			
Mode	Experimental Frequency (±1 Hz)	Theoretical Frequency (Hz)	Difference (%)
(1, 0)	74	102	32
(0, 0)	126	131	4
(2, 0)	181	190	5
(3, 0)	416	429	3
(4, 0)	727	750	3
(0, 1)	–	772	–
(1, 1)	771	858	11
(5, 0)	1112	1144	3
(2, 1)	1146	1203	5
(6, 0)	1568	1611	3
(3, 1)	1717	1774	3
(7, 0)	2052	2147	5
(0, 2)	2096	2276	8
(1, 2)	2160	2412	11
(4, 1)	2382	2458	3
(8, 0)	2692	2751	2
(2, 2)	2743	2907	6
(5, 1)	3131	3233	3
(9, 0)	3358	3420	2
(3, 2)	3607	3741	4
(6, 1)	3964	4089	3
(10, 0)	4091	4154	2

with the inner boundary being defined by the diameter of the washers. We note, however, that the arrangement did not completely restrict shear motion of the plate at the inner boundary.

Although interferometry is the most effective method of identifying the resonant frequencies and their associated modal patterns, two of the modes were heavily damped and difficult to excite acoustically. To identify modes that were heavily damped, the sound produced by striking the plate was recorded and a power spectrum was produced. The peaks in the power spectrum that could not be identified by interferometry were compared to a finite-element (FE) model of the plate produced in the commercially available program Femap, a FE and postprocessing program developed by Siemens PLM Software. A list of the first 22 modes and their frequencies is shown in the second column of Table I. In this table the mode structure is annotated as (m, n), where m is the number of nodal diameters and n is the number of radial nodes. Of the two heavily damped modes, the (7, 0) mode was identified from the power spectrum but the (0,1) mode was so heavily damped that it could not be identified acoustically or interferometrically.

III. THEORY

Having experimentally determined a large number of the normal mode frequencies and modal shapes of the annular plate, we utilized CTP theory to predict the resonance frequencies of the plate. One would expect that since the ratio of the plate thickness to the diameter of the plate is approximately and order of magnitude smaller than what is normally considered the limit of being thin, CTP theory should accurately predict the resonant frequencies of the lower modes of vibration. Thus the fact that shear motion may occur at the inner boundary should be insignificant to the analysis.

The fundamental assumptions of CTP theory are that the magnitude of the vibrations of the plate are small and the plate is thin enough that the flexural stress perpendicular to the middle plane

TABLE II. Parameters used to model the annular thin plate. The parameters *a* and *b* are the inner and outer radii, respectively. The other parameters are defined in the text.

Parameters used in the model	
a	6.25 mm
b	128.3 mm
h	2 mm
σ	5.32 kg/m^2
E	9.0×10^{10} N/m^2
ν	0.22
R	1.0×10^{-5} kg/m^{2}s

is negligible. These assumptions effectively reduce the analysis of the three-dimensional plate to a two-dimensional problem.[13] The material of the plate is also assumed to be elastic, homogeneous and isotropic, as well as initially flat. These conditions are all met in the experiments under consideration here.

In cylindrical coordinates, CTP theory describes the transverse motion of a thin plate by the differential equation

$$-D\nabla^4 w(r,\phi,t) + p(r_0,\phi_0,t) - R\frac{\partial w(r,\phi,t)}{\partial t} = \sigma\frac{\partial^2 w(r,\phi,t)}{\partial t^2}, \quad (1)$$

where w is the displacement of the plate, R is a damping coefficient, σ is the mass per unit area, and $p(r_0, \phi_0, t)$ is the applied force per unit area. The coefficient D is known as the flexural rigidity and is defined as

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)}, \quad (2)$$

where E is the elastic modulus of the plate, h is the thickness of the plate, and ν is Poisson's ratio.

To compare the data presented in Section II with predictions from CTP theory, Eq. (1) was numerically integrated using a finite difference computer program that was described in detail previously by Moore, *et al.*, which was validated by comparing the predicted frequencies with those determined analytically; the average variation was approximately 2%.[5] The plate was assumed to be clamped on the inner boundary, which was defined by the diameter of the washers securing the plate, and the outer edge was assumed to be free. The parameters used in the model are shown in Table II. Note that the damping coefficient R was very small and investigations showed that damping at this level had no effect on the predicted frequencies.

In the program, the plate was divided into 200 radial and 120 angular points (24,000 total points), with the displacement of each of these points being given by the finite difference solution to Eq. (1). Two different driving scenarios were applied to ensure that the resonant frequencies were properly identified. First, the plate was modeled as being struck with a pulse of short duration (5.0×10^{-4} s) and, using a time step of 6.0×10^{-9} s, the program was run to simulate a total time of 1.0 s. The resonant frequencies were then determined by performing a Fourier transform on the displacement at a single point as a function of time. Then, to identify each mode of vibration, a sinusoidal force at each resonant frequency was applied to the plate and the program was run for a sufficiently long time to establish a clear modal pattern. Plots of the displacements as a function of time and position allowed for the unambiguous identification of each mode.

The experimentally determined resonance frequencies are compared to those predicted by CTP theory in Table I and shown graphically in Fig. 2. Note that the theoretical predictions are in good agreement with the experimental data for all modes except the (1, n) modes, with a difference of approximately 32% in the case of the (1, 0) mode.

The validity of the predictions of the finite difference model was further investigated by modeling the plate using the commercially available FE modeling program Femap. This FE model does not include any of the assumptions concerning stress, strain, or bending and twisting moments that are

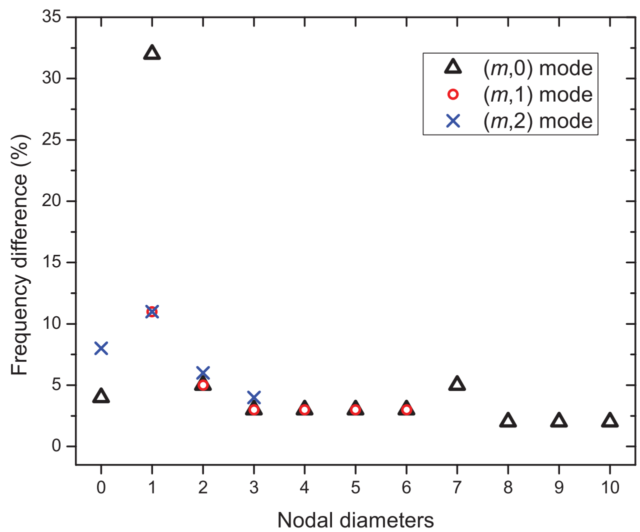

FIG. 2. Plot of the difference between the measured resonance frequencies and those predicted by CTP theory. The data are shown in Table I.

TABLE III. Predicted resonance frequencies from both CTP theory and a FE model assuming the inner boundary is clamped in all directions.

Mode	Frequencies of Normal Modes CTP Theory (Hz)	FE Model (Hz)	Difference (%)
(1, 0)	102	94	9
(0, 0)	131	129	2
(2, 0)	190	188	1
(3, 0)	429	430	1
(4, 0)	750	752	1
(0, 1)	772	746	3
(1, 1)	858	825	4
(5, 0)	1144	1150	1
(2, 1)	1203	1189	1
(6, 0)	1611	1623	1
(3, 1)	1774	1773	1
(7, 0)	2147	2171	1
(0, 2)	2276	2195	4
(1, 2)	2412	2323	4
(4, 1)	2458	2464	1
(8, 0)	2751	2792	1
(2, 2)	2907	2858	2
(5, 1)	3233	3246	1
(9, 0)	3402	3486	2
(3, 2)	3741	3739	1
(6, 1)	4089	4116	1
(10, 0)	4154	4254	2

included in CTP theory. Instead, it solves the full three-dimensional elastic problem. The FE model divided the plate into 11,892 three dimensional elements with two volume elements spanning the thickness of the plate.

Table III shows a comparison between the resonance frequencies predicted by CTP theory and those predicted by the FE model, with the assumption that the inner boundary is clamped in all dimensions. The average difference between the two sets of frequencies is only 2%, which is within the uncertainty in the finite difference method used to make the CTP theory predictions.[5] We note that the largest differences occur for the (0, n) and (1, n) modes, but with the exception of the (1, 0) mode, these are all less than or equal to 4%. The excellent agreement seen in Table III between the predicted resonance frequencies from CTP theory and the FE model with this inner boundary condition is an important result. The similarity between the predicted resonant frequencies for the two models, one two-dimensional and one three-dimensional, seems to further support the validity of CTP theory for describing the motion of this plate to quite high frequencies. That is, when the inner boundary is fixed in all dimensions, assuming that the plate is essentially two-dimensional appears to be a valid approximation since solving a fully three-dimensional problem yields results in overall excellent agreement. Thus, we are left with the question of why there are significant differences between the experimentally determined and theoretically predicted resonance frequencies of the (1, n) modes, with a specific interest in the lowest mode of vibration (1,0).

IV. INVESTIGATING BOUNDARY CONDITIONS

The observed difference between the experimentally determined resonance frequency of the lowest normal mode and that predicted using CTP theory indicates the possibility that the system being modeled does not reflect the physical reality of the vibrating flat plate. Therefore, we investigated the effect that varying the boundary conditions has on the resonance frequencies predicted by the FE model.

In the analysis presented above it was assumed that the inner diameter of the annular plate is fixed, as one might expect given the experimental arrangement described in Sec. II. Although the screw securing the inner boundary did not make firm contact with the interior of the plate, the plate was securely prevented from any out-of-plane motion by the presence of the washers, which were securely in contact on both sides of the plate and the support structure. Since it is commonly assumed that shear strains can be ignored in determining the resonant frequencies for the low modes of vibrations in plates with a thickness comparable to that used in this experiment, one may conclude that the effects of in-plane motion of the inner boundary are negligible for these modes, which is the assumption used to derive Eq. (1). However, especially given the large discrepancies between the experimentally measured resonance frequency for the (1,0) mode and the predictions of both CTP theory and the FE model for this mode, it is reasonable to suspect that the in-plane motion of the inner boundary may be important.

To investigate this possibility, the FE model was altered so that the restriction on the in-plane motion at the inner boundary was removed, while keeping the out-of-plane motion constrained. Thus the model was adjusted so that it accurately reflected the actual boundary conditions of the plate. Upon relaxing this condition, there was a significant change in the predicted resonance frequencies of the (0, n) and (1, n) modes, with the most significant being the frequency of the (1, 0) mode. Table IV includes a list of the experimentally determined resonance frequencies of the annular plate compared to the predicted frequencies from the FE model with the inner boundary condition restricting the in-plane motion removed. These results clearly demonstrate that allowing for in-plane motion of the clamped region results in predicted resonance frequencies consistent with those found experimentally, and in the process verifies the validity of the FE model. In contrast to the case where the in-plane motion was restricted, the predicted resonance frequencies are almost all within 4% of those determined experimentally and the predicted frequencies of the (1, n) modes are all within 2% of the measured values.

V. DISCUSSION

The results shown in Table I and Fig. 2 indicate that CTP theory adequately predicts many of the first 22 resonance frequencies of a thin annular plate clamped at the center; however, noticeable discrepancies do exist between experimental results and the predictions of CTP theory for the (1, n) modes, especially at the lowest resonant frequency. These noticeable differences also exist between

TABLE IV. Measured frequencies at which the normal modes of an annular flat plate occur beside the theoretical predictions of the FE model when in-plane shear motion at the inner boundary is allowed.

	Frequencies of Normal Modes		
Mode	Experimental Frequency ($\pm$1 Hz)	Theoretical Frequency (Hz)	Difference (%)
(1, 0)	74	72	2
(0, 0)	126	120	5
(2, 0)	181	187	3
(3, 0)	416	430	3
(4, 0)	727	752	3
(0, 1)	–	678	–
(1, 1)	771	762	1
(5, 0)	1112	1150	3
(2, 1)	1146	1178	3
(6, 0)	1568	1623	3
(3, 1)	1717	1773	3
(7, 0)	2052	2171	6
(0, 2)	2096	2015	4
(1, 2)	2160	2173	1
(4, 1)	2382	2464	3
(8, 0)	2692	2791	4
(2, 2)	2743	2821	3
(5, 1)	3131	3246	4
(9, 0)	3358	3486	4
(3, 2)	3607	3737	4
(6, 1)	3964	4116	4
(10, 0)	4091	4254	4

the experimentally determined resonance frequencies and those predicted using the commercially available FE software, unless in-plane motion at the inner boundary is allowed.

In comparing Tables I and III with Table IV, it is clear that the (0, n) and (1, n) modes are the most affected by relaxing the in-plane boundary condition, whereas the frequencies of the other modes are not significantly changed. This is an interesting result, since it implies that these modes are the most affected by a shear deformation of the clamped region. In fact, the FE model predicts a noticeable decrease in the in-plane motion with an increase in the number of nodal diameters of the normal mode patterns. It appears that for mode patterns with a higher number of nodal diameters, opposing in-plane stresses that are in close proximity reduce the in-plane motion of the plate. This is not the case for modes with no nodal diameter or for modes with only one nodal diameter. In these cases shear deformation of the interior boundary does not change direction ((0, n) modes), or changes direction at only two points ((1, n) modes). Therefore, the effects of shear deformation are more pronounced for these modes.

VI. CONCLUSIONS

The work reported here shows that although CTP theory is widely accepted as sufficient for analyzing the lowest normal modes of thin annular plates, there are important cases where the approximations involved in developing the theory are not valid. Despite the fact that the annular plate used in the experiments described here meets the conditions under which CTP theory is considered an acceptable means for modeling the motion of low frequency modes of vibration, the lowest resonance frequency is not accurately predicted. Furthermore, the accuracy of the prediction is significantly improved by relaxing the restriction on in-plane motion at the inner boundary, indicating that, contrary to what is widely believed, the effects of shear strain can be important in thin plates at low frequencies.That is, CTP theory is not valid despite the fact that the dimensions of the plate fall well within the limitations under which it is normally applied.

From this work one can conclude that, although it is a common assumption found in the literature as well as recent textbooks,[14] it is not appropriate to use CTP theory when modeling thin annular plates unless there is no possibility of shear deformation at the internal boundary. While this condition is often assumed, and it is relatively easy to ensure that there is no out-of-plane motion at the inner boundary of an annular plate, in an actual physical system it is very difficult to ensure that there is no in-plane motion.Yet shear deformation, which is normally considered negligible in thin plates, clearly affects the resonance frequencies of some of the lowest modes.

Since the effect of shear deformation on the resonance frequencies of thin annular plates is not negligible, one may suspect that there are other thin-plate geometries where in-plane motion is important. Indeed, it is reasonable to assume that such effects will be observable in any instance where the modal pattern produces large areas of the internal boundary without opposing in-plane forces.

From Table I, it is clear that resonant frequencies predicted by CTP theory agree well with those measured experimentally for both the $(m, 0)$ and $(m, 1)$ modes with $m > 1$ up to over 4K Hz. Thus, there is no indication that CTP theory fails to accurately predict the resonance frequencies in very thin plates for many of the modes of vibration, even when the in-plane motion at the inner boundary cannot be restricted. Finally, as indicated by the comparison between CTP theory and the three-dimensional model in Table III, if such motion could be restricted experimentally then CTP theory should accurately predict the resonance frequencies of even the lowest modes of vibration. However, contrary to what is commonly stated in the literature, when in-plane motion is not restricted one cannot assume that CTP theory is applicable simply because the plate is thin.

We note in closing that it is not clear that this restriction on the use of CTP theory applies to plates where the ratio of inner to outer radii is not small. Further work is required to verify that that this restriction holds under these conditions.

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