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Cayley Map Embeddings of Complete Graphs With Even Order

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1. Abstract

German mathematician Claus Michael Ringel used voltage graphs to embed complete graphs onto orientable surfaces such that none of the graph's edges cross each other. Cayley maps do the same whilst being simpler to work with. The goal is to determine the efficiency of Cayley maps in embedding complete graphs onto orientable surfaces. This article focus on complete graphs of even order with an emphasis on graphs whose orders are congruent to 6 modulo 12 and 0 modulo 12. We establish 12 distinct classes that each have their own unique qualities. Through the generalization of a previous technique, we prove a nontrivial bound on the Cayley genus of graphs whose order is congruent to 6 modulo 12. We also show that Cayley maps cannot embed a complete graph onto its optimal genus for 8 out of the 12 classes provided the graph's order is greater than 6.

Keywords: Cayley map, Cayley genus, complete graph embeddings, Claus Ringel, J.W.T. Youngs

2. Introduction

A research topic in the area of topological graph theory is how to embed complete graphs onto orientable surfaces such that none of the graph's edges cross each other. One application that has been brought up involves the construction of a computer chip [2]. To prevent the chip from short-circuiting, one needs to ensure that none of its wires are crossing. In order to achieve this for a chip with many wires, one must drill holes through the chip's surface, or add layers to the chip. The more wires there are, the more holes that need to be drilled, and the less obvious it becomes as to what the actually wiring itself should look like on the chip. Cayley maps can be used to help solve this problem. A Cayley map embedding provides a cyclical edge rotation that can be followed to map the wires on the surface of the chip such that no edges cross. While Ringel's voltage graphs were also effective in doing so, they used complex mathematical techniques, while Cayley maps deploy relatively simple methods. That being said, it is not always possible for Cayley maps to embed graphs onto an orientable surface of their optimal genus. The purpose of this paper is to explore how effective Cayley maps are at embedding complete graphs onto orientable surfaces, particularly when the graph is of an even order.

Miriam Schleinblum has previously researched the same topic. In her paper, she focused on graphs of a prime order, and primarily worked under the group $\mathbb{Z}_n$ [6]. She proposed a conjecture that stated if a graph's order is congruent to 7 (mod 12), then Cayley maps can be used to embed the graph onto an orientable surface of its genus. She noticed this because many of the numbers which are congruent to 7 (mod 12) are prime, and she was able to see the pattern that was occurring. Schleinblum later developed a program that was able to create Cayley map embeddings to see how well $\mathbb{Z}_n$ did at embedding graphs when their order was congruent to 7 (mod 12). The computer was able to embed graphs with up to 115 vertices which provides enticing yet inconclusive evidence towards her conjecture. This paper sorts out why graphs of an order congruent to 7 (mod 12) are special, and outlines other distinct classes of graphs that hold its same unique characteristics.

A year later, Hannah Hendrickson did research on creating triangular faces in the embedding to maximize the number of faces the embedding produces, which minimizes the embedding's genus. In her article, Hendrickson tackles the issue of ensuring a Cayley map's rotation of edges is a cycle. She invented a new structure called a $G_{\mathcal{P}}$ graph, which stands for group partition graph. In her paper, she proved that if the $G_{\mathcal{P}}$ graph of a Cayley map embedding has a non-backtracking Eulerian tour, then the rotation of edges it produces truly is a cycle [3]. This combined with a technique called graph reduction helps immensely in formulating valid Cayley map embeddings. This thesis will show how her techniques can be used to potentially create optimal Cayley maps for an entire family of complete graphs.

In Section 3, we will establish the definitions and notation needed to understand Cayley map embedding, and give an example of a Cayley map embedding. In Section 4, we set up the 12 distinct classes of Cayley maps based on the order of the complete graph at hand. In Section 5, we outline a strategy on how to immediately disqualify Cayley map embeddings, and prove theorems that come as a consequence of these disqualification techniques. Most notably, Section 5 contains a Theorem that is an absorption of Schleinblum's 12,7 Conjecture. Theorems for complete graphs of an even order are outlined in Section 6, along with a non-trivial bound on the Cayley genus when the complete graphs order is congruent to 6 (mod 12). Section 7 contains a Conjecture about the non-trivial bound provided in Section 6, and proposes methods on how to potentially prove said Conjecture. Lastly, Section 8 will summarize the implications of the research and establish new questions in the research area that future research pupils can work on.

3. Definitions and Notation

This Section will introduce fundamental topics in topological graph theory and abstract algebra. Understanding these concepts is pertinent towards conceptualizing Cayley map embedding. Around the end of the Section, Cayley map embedding will be explained by combining the definitions outlined in this Section. Two examples of Cayley map embedding will be provided in order to give the reader some intuition of the technique.

Definition 1. *We say n is congruent to r modulo d (denoted $n \equiv r \pmod{d}$) if and only if $n = dk + r$ for some $k \in \mathbb{Z}$.*

Definition 2. *A set, G, with a binary operation, $*$, is a group (denoted $(G, *)$, or just G) if...*

(1) *There exists an identity element in the set (there exists an $e \in G$ such that for all $g \in G$, $g * e = e * g = g$).*

(2) *The set is closed under inverses (for all $g \in G$, there exists some $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$).*

(3) *The operation is associative (for all $g, h, f \in G$, $(g * h) * f = g * (h * f)$).*

Definition 3. *Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$.*

Definition 4. *Let $+_n$ be the binary operation on $\mathbb{Z}_n$ such that $g +_n h = k$ if and only if $g + h \equiv k \pmod{n}$.*

Example 5. *$(\mathbb{Z}_n, +_n)$ is a group, and is typically denoted as just $\mathbb{Z}_n$.*

Example 6. *$(\mathbb{Z}_n \times \mathbb{Z}_m, +)$ is a group where if $g_1, g_2 \in \mathbb{Z}_n$ and $h_1, h_2 \in \mathbb{Z}_m$, then $(g_1, h_1) + (g_2, h_2) = (g_2 +_n h_1, h_1 +_m h_2)$.*

Definition 7. *A graph, $G = (V, E)$ is comprised of a vertex set V and an edge set E. Each edge is associated to either one or two vertices, called the endpoints of the edge.*

Definition 8. *The order of a graph denotes the cardinality of its vertex set. The size of a graph denotes the cardinality of its edge set.*

Definition 9. *Two vertices are adjacent if and only if they are both associated to a single edge.*

Definition 10. *The complete graph, K_n, has n vertices where each vertex is adjacent to every other vertex. K_n has order n and size $\frac{n(n-1)}{2}$.*

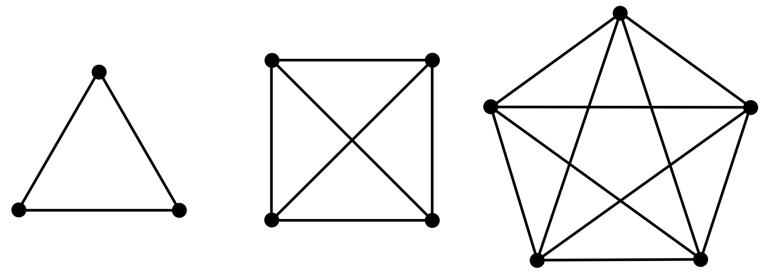

Figure 1. K_3, K_4, and K_5

The next part of this Section will be devoted towards describing orientable surfaces and how graphs are embedded onto them. An orientable surface is one in which clockwise rotation is well-defined. On some surfaces, such as a Möbius strip, one can move an object along a particular path on the surface and have the object end up at the same point it started at but it comes back as its mirrored image. This cannot happen on a sphere, which is an example of an orientable surface. In 1866, Camille Jordan proved that the sphere, torus, double-torus, and so on make up all orientable surfaces up to topologically homeomorphism [4]. These surfaces are characterized by the number of holes they have. The number of holes an orientable surface has is called its genus.

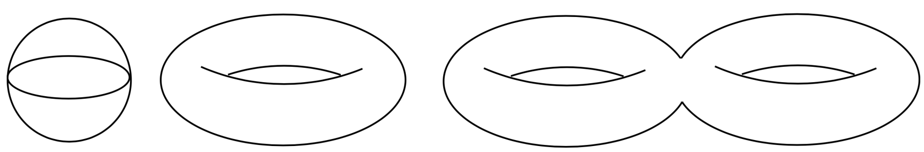

Figure 2. The sphere, torus, and double-torus

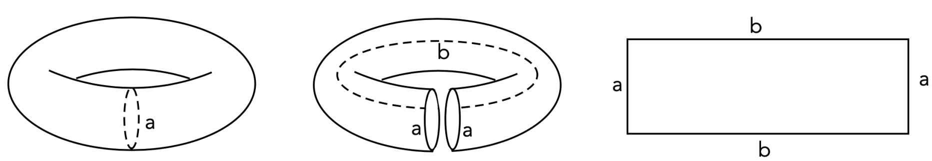

Figure 3. The flat torus

Definition 11. *An embedding of a graph $G = (V, E)$ onto a surface S consists of...*

(1) *A one-to-one function $f_V : V \to S$.*

(2) *A continuous, one-to-one function* $f_e : [0,1] \to S$ *for each* $e \in E$*, such that if* v_0 *and* v_1 *are endpoints, then* $f_e(0) = v_0$ *and* $v_e(1) = v_1$ *(or vice versa) with the property that* $f_{e_1}(x) = f_{e_2}(y)$ *for any* $x, y \in (0,1)$ *implies* $e_1 = e_2$ *(and* $x = y$*).*

The larger the size of the graph, the greater the orientable surface's genus must be in order to embed the graph onto its surface without having any edges cross. The minimum genus a graph, G, needs in order to embed onto an orientable surface is called the graph's genus, and is denoted as $\gamma(G)$. In 1968, Claus Ringel and J.W.T. Youngs proved a formula for the genus of a complete graph [5]. An optimal embedding embeds the graph onto a surface of the graph's genus. A two-cell embedding is an embedding for which after mapping vertices and edges onto an orientable surface, faces are formed that are each homeomorphic to an open disk in $\mathbb{R}^2$. From now on, any time the word "embedding" is used, assume that it is a two-cell embedding. The faces formed by a two-cell embedding is designated F.

Definition 12. (*Ringel's Theorem*) *The genus of a complete graph is*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

Theorem 13. *For all* $n \in \mathbb{N}$ *and* $n \geq 3$, $\gamma(K_{n+12}) = \gamma(K_n) + 2n + 5$.

Proof. According to Ringel's Theorem,

$$\begin{aligned}\gamma(K_{n+12}) &= \left\lceil \frac{((n-3)+12)((n-4)+12)}{12} \right\rceil \\ &= \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil + 2n + 5 \\ &= \gamma(K_n) + 2n + 5.\end{aligned}$$

$\square$

Definition 14. (*Euler characteristic formula*) *The Euler characteristic,* χ*, of an embedding is*

$$\chi = |V| - |E| + |F|,$$

where the genus, g*, of the surface is*

$$g = \frac{2-\chi}{2}.$$

Swiss mathematician Leonhard Euler showed that that every graph embedded in the plane or on a sphere has an Euler characteristic of 2 [1]. It was later proven that the Euler characteristic is an invariant of any two-cell embedding on a surface given the surface's genus.

Definition 15. *Suppose H is a group with n elements and X is a subset of $H - \{e\}$ that is closed with respect to inverses. The Cayley graph $C_G(H, X)$ is a graph on n vertices, labeled by the n elements of H. The edges are determined by X : vertices u and v are adjacent if and only if there exists some $x \in X$ such that $u = v * x$.*

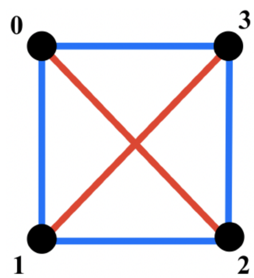

FIGURE 4. $C_G(\mathbb{Z}_4, \{1, 2, 3\})$

Figure 4 shows that K_4 can be represented as the Cayley graph $C_G(\mathbb{Z}_4, \{1, 2, 3\})$, where each of the edges are determined by the non-zero elements of $\mathbb{Z}_4$. For instance, start at 0. Notice that $1 \in X$ and $1 = 0 +_4 1$. According to Definition 15, this means that 0 and 1 must be adjacent. The same is true for 1 and 2, 2 and 3, and 3 and 0. Hence, the blue edges are generated by 1. One could also say that 3 is generating the blue edges too. Lastly, the red edges are generated by 2, since $3 = 1 +_4 2$ and $2 = 0 +_4 2$. Thus, all the vertices will be adjacent to one another since each vertex can reach any other vertex by the addition of some element in X. This ensures that the graph is complete. However, notice how Figure 4 has two edges which cross each other. Cayley maps are used to embed Cayley graphs onto orientable surfaces without having edges cross.

Definition 16. *The Cayley map $C_M(H, \rho)$ embeds Cayley graph $C_G(H, X)$ onto a surface, where X is a subset of $H - \{e\}$ that is closed with respect to inverses and $\rho = (x_1, x_2, \ldots, x_k)$ is a cyclic permutation of X. The Cayley graph $C_G(H, X)$ is embedded in a surface so that ρ gives the counterclockwise rotation of edges around each vertex in the embedding.*

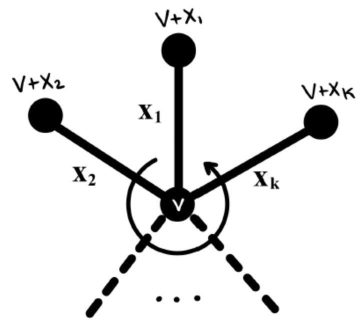

FIGURE 5. The counterclockwise rotation of edges around each vertex of a particular Cayley map

Figure 5 illustrates the counterclockwise rotation of edges around each vertex of a Cayley map with $\rho = (x_1, x_2, \ldots, x_k)$. As an example, we will go back to K_4 with $C_M(\mathbb{Z}_4, (1, 3, 2))$ as its Cayley map.

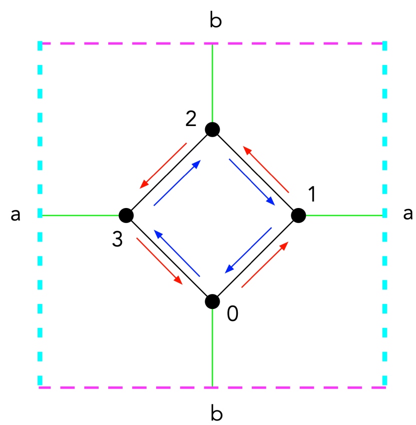

FIGURE 6. $C_M(\mathbb{Z}_4, (1, 3, 2))$ on a flat torus

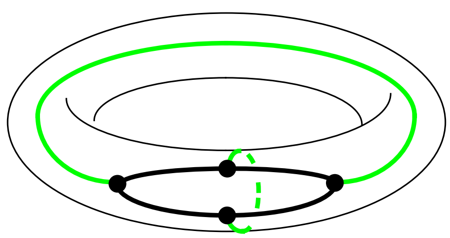

FIGURE 7. $C_M(\mathbb{Z}_4, (1, 3, 2))$ embedded on a torus

In Figure 6, a rectangle is being used to represent a torus. Figure 6 shows that $\rho = (1, 3, 2)$ determines the counterclockwise rotation of edges around any particular vertex $v \in \mathbb{Z}_4$. By adding non-zero elements to any vertex, we can reach each of the other vertices in order to make the graph complete. For example, start at the vertex 1. Notice that there is a red arrow that creates an edge between 1 and 2 by adding $1 \in X$ to 1. Now, note the next edge pointing out from 1 in the counterclockwise direction. This edge is adding $3 \in X$ to 1 to make vertices 1 and 0 adjacent. The last edge in the counterclockwise rotation is adding $2 \in X$ to 1 to produce an edge around the torus to the vertex 3. The reason why this edge has no directed arrows is because 2 is its own inverse under $\mathbb{Z}_4$. Hence, $\rho = (1, 3, 2)$ provides the order of the counterclockwise rotation of edges around every vertex in $C_M(\mathbb{Z}_4, (1, 3, 2))$.

The edges of the embedding come together to form faces. From Figure 7, we can see that the number of vertices, edges, and faces is 4, 6, and 2 respectively. From Definition 14, $\chi = |V| - |E| + |F| = 4 - 6 + 2 = 0$, and the genus of the embedding is $g = \dfrac{2-\chi}{2} = 1$. Hence, $C_M(\mathbb{Z}_4, (1, 3, 2))$ is embedded on a torus. Since $g \neq \gamma(K_n)$, the genus of $C_M(\mathbb{Z}_4, (1, 3, 2))$ does not match the genus of K_4, making the Cayley map embedding's genus suboptimal. Later on in the paper, it will be proven that Cayley map $C_M(\mathbb{Z}_{12k+4}, \rho)$ never embeds K_{12k+4} onto its genus. Next, we will use Cayley map $C_M(\mathbb{Z}_2 \times \mathbb{Z}_2, (0, 1), (1, 0), (1, 1))$ to optimally embed K_4 onto a sphere.

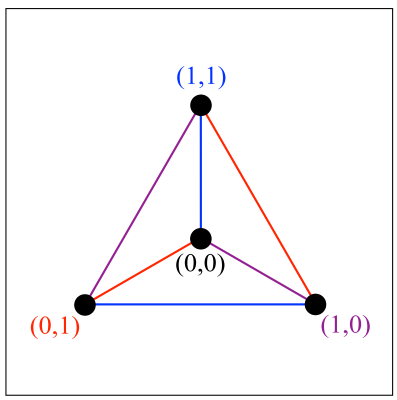

Figure 8. $C_M(\mathbb{Z}_2 \times \mathbb{Z}_2, ((0, 1), (1, 0), (1, 1)))$ embedded on a sphere

In Figure 8, a square is being used to represent a sphere. This square can also be viewed as a section of a sphere, with the Cayley map being embedded onto that section. Notice that Figure 8 has no directed arrows since all the elements in $X = \{(0, 1), (1, 0), (1, 1)\}$ are their own inverses under $\mathbb{Z}_2 \times \mathbb{Z}_2$. Each element of X is color-coded along with the edges that said element creates in the embedding. Again, the edges formed around each vertex are generated by the counterclockwise rotation specified in ρ. One can see that the embedding creates 4 faces. Therefore, the embedding has an Euler characteristic $\chi = 4 - 6 + 4 = 2$. Hence, $g = \dfrac{2-\chi}{2} = 0$, which tells us that the embedding is on a sphere. Going back to Ringel's Theorem, $\gamma(K_4) = 0$, so the best we can do is map K_4 on a sphere. Since $g = \gamma(K_4)$, Cayley map $C_M(\mathbb{Z}_2 \times \mathbb{Z}_2, ((0, 1), (1, 0), (1, 1)))$ is an optimal embedding of K_4, and $\gamma_c(K_4) = \gamma(K_4)$.

Another way to describe the faces of a Cayley map is by looking at λ, which is a permutation of disjoint cyclic permutations where each factor in λ describes a face type. For example, Figure 6 has 2 face types. One face type is generated by the blue arrows (corresponding to 3 in X) and the other face type is generated by a mixture of the red and green arrows (corresponding to 1 in X and 2 in X respectively). Therefore, there should be two disjoint cyclic permutations in λ that describe these 2 face types. The following definition outlines the relationship between ρ and λ, and will be used to calculate the λ of $C_M(\mathbb{Z}_4, (1, 3, 2))$ after a brief explanation of the surrounding notation and nature of λ.

Definition 17. *Suppose H is a group and X is a subset of H that is closed with respect to inverses. Then $\lambda(x) = \rho(x^{-1})$ (and consequently, $\rho(x) = \lambda(x^{-1})$).*

If $j = (x_1 x_2 \dots x_n)$ is a cyclic permutation, then $|j|$ denotes the number of elements in j. So $|j| = n$ in this example. We also define the multiplicity of j, denoted $mult(j)$, to be the order of $x_1 * x_2 * \cdots * x_n$ in the group of the embedding X. That is, $mult(j) = m$ where m is the smallest positive integer such that $(x_1 * x_2 * \cdots * x_n)^m = e$. When writing λ, we write $\lambda = \lambda_1 \cdot \lambda_2 \cdots \lambda_m$ as a product of m disjoint cyclic permutations. The face length of λ_i, denoted $FL(\lambda_i)$, is $|\lambda_i| \cdot mult(\lambda_i)$, and represents the number of sides of the polygon generated by λ_i. The number of faces produced by λ_i can then be found by deploying the formula $FN(\lambda_i) = \dfrac{n|\lambda_i|}{FL(\lambda_i)} = \dfrac{n}{mult(\lambda_i)}$, where n is the number of vertices. This equation is significant since the goal of this paper is to maximize the number of faces each λ_i produces in order to augment the Euler characteristic of the Cayley map embedding, which minimizes the genus of the surface.

Going back to Figure 6 where $\rho = (1, 3, 2)$, we can use Definition 17 to calculate λ and show how each λ_i describes a face type. To start, $\lambda(1) = \rho(1^{-1}) = \rho(3) = 2$. Next, $\lambda(2) = \rho(2^{-1}) = \rho(2) = 1$, so λ_1 is $(1, 2)$. Lastly, $\lambda(3) = \rho(3^{-1}) = \rho(1) = 3$, so 3 maps back to itself, making λ_2 equal to (3). The cyclic permutation, λ_2, describes the face type in Figure 6 that is created by the blue arrows. Next, λ_1 describes the face formulated by the green and red arrows since one can alternate traveling along the red and green arrows and end up back at their original vertex. To calculate the number of faces $CM(\mathbb{Z}_4, (1, 3, 2))$ produces, we have to deploy the face number formula, $FN(\lambda_i)$, on each λ_i and sum them all together. For λ_1, $FN(\lambda_1) = \dfrac{n}{mult(\lambda_1)} = \dfrac{4}{ord(1+2)} = 1$. Similarly, for λ_2, $FN(\lambda_2) = \dfrac{n}{mult(\lambda_2)} = \dfrac{4}{ord(3)} = 1$. This gives us a total of 2 faces, which is exactly what is observed in Figures 6 and 7. The next definition will describe the average number of faces an element produces.

Definition 18. *If* $x \in \lambda_i$*, then . . .*

$$AF(x) = \frac{FN(\lambda_i)}{|\lambda_i|}.$$

If $\lambda_i = (x)$ is a factor of λ with multiplicity 3, then $FN(\lambda_i) = \frac{n}{3}$, and $AF(x) = \frac{n/3}{1} = \frac{n}{3}$. Hence, the single element x produces on average $\frac{n}{3}$ faces. If $\lambda_i = (x, y, z)$ is a factor of λ with length 3 and multiplicity 1, then $FN(\lambda_i) = \frac{n}{1} = n$, and $AF(x) = AF(y) = AF(z) = \frac{n}{3}$. Ergo, each of the 3 elements x, y, and z produces on average $\frac{n}{3}$ faces. The following Theorems prove that no element in λ can produce n or $\frac{n}{2}$ faces.

Theorem 19. *For all* $x \in \lambda$*,* $AF(x) \neq n$*.*

Proof. Suppose the opposite is true, and let λ_i be the particular factor of λ this x belongs to. Then,

$$\begin{aligned}
\frac{FN(\lambda_i)}{|\lambda_i|} &= n \\
\frac{n}{mult(\lambda_i)|\lambda_i|} &= n \\
\frac{1}{mult(\lambda_i)} &= |\lambda_i|.
\end{aligned}$$

Therefore, since $mult(\lambda_i), |\lambda_i| \in \mathbb{N}$, $mult(\lambda_i) = 1$, and $|\lambda_i| = 1$. This means $\lambda_i = (e)$, but (e) can't be a factor of λ since $e \notin X$, which is a contradiction. Hence, $AF(x) \neq n$. $\square$

Theorem 20. *For all* $x \in \lambda$*,* $AF(x) \neq \frac{n}{2}$*.*

Proof. Again, suppose the opposite is true, and let λ_i be the particular factor of λ this x belongs to. Then,

$$\begin{aligned}
\frac{FN(\lambda_i)}{|\lambda_i|} &= \frac{n}{2} \\
\frac{n}{mult(\lambda_i)|\lambda_i|} &= \frac{n}{2} \\
\frac{2}{mult(\lambda_i)} &= |\lambda_i|.
\end{aligned}$$

Therefore, since $mult(\lambda_i), |\lambda_i| \in \mathbb{N}$, either $mult(\lambda_i) = 1$ and $|\lambda_i| = 2$, or $mult(\lambda_i) = 2$ and $|\lambda_i| = 1$. Assume the former is the case. Then, we know that $\lambda_i = (x, x^{-1})$ for some $x \in X$. Therefore, according to Definition 17, $\rho(x) = \lambda(x^{-1}) = x$. Hence, ρ contains a factor (x). Thus, ρ is not a cyclic permutation and would not create a Cayley map. Now, assume $mult(\lambda_i) = 2$ and $|\lambda_i| = 1$. We know that $\lambda_i = (x)$ where x is its own inverse under the group since $mult(\lambda_i) = 2$. Ergo, $x = x^{-1}$. Therefore, $\rho(x) = \lambda(x^{-1}) = \lambda(x) = x$. Thus, Similarily to the last case, ρ would not be a cycle, and a Cayley map would not be produced. Hence, $AF(x) \neq \frac{n}{2}$. □

Definition 21. *Let the number of elements in X that produce on average $\frac{n}{i}$ faces be denoted as m_i.*

Definition 22. *Let $x \in X$. If $AF(x) = \frac{n}{3}$, then let x be called maximum face-generating. Otherwise, the element is submaximum face-generating.*

Theorem 23. *(The Cayley map face formula) The number of faces a Cayley map embedding produces is*

$$|F| = \sum_{i=3}^{\infty} \frac{n}{i} m_i = \sum_{x \in X} AF(x).$$

Definition 24. *The minimum genus that can be produced by a Cayley map embedding of a graph G is $\gamma_c(G)$.*

Definition 25. *The minimum genus that can be produced by a Cayley map of a graph G using a cyclic group is $\gamma_c^{\circlearrowleft}(G)$.*

Observation 26. (*The trivial Cayley map bound*) *For all $n \in \mathbb{N}$, $\gamma_c(K_n) \geq \gamma(K_n)$.*

4. The 12 Classes

Given there are different elements that create a different number of faces, it is important to write the number of faces needed to be produced in terms of these m_i's in order to figure out which groups are good to use and which are not. Let $F_\gamma(n)$ and $F_{\gamma_c}(n)$ be the number of faces required to produce K_n's genus and Cayley genus respectively. Similarily, let $\chi_\gamma(n)$ and $\chi_{\gamma_c}(n)$ be the Euler characteristic required to produce K_n's genus and Cayley genus respectively. Recalling the previously stated goal to maximize the number of faces a Cayley map embedding produces to minimize its genus, one might assume that it is preferable to have all of the elements in X be maximum face-generating. To calculate how many faces would be created, think back to the fact that there are always $n-1$ elements in X for the Cayley embedding of K_n. Therefore, to have the Cayley map embedding produce as many faces as possible, we would want all $n-1$ elements in X to produce on average $\frac{n}{3}$ faces. In other words, we would have $m_3 = n-1$. Now, since every element in X is maximum face-generating, we know that $m_i = 0$ for all $i \geq 4$. Hence, according to Definition 23, $F_{\gamma_c}(n) = \frac{n}{3}(n-1)$. While this goal to have all the elements of λ be maximum face-generating is logical, it is oftentimes *impossible*. In fact, Theorem 27 and Theorem 28 prove that $F_{\gamma_c}(n)$ has to be less than $\frac{n}{3}(n-1)$ for most cases.

Theorem 27. *Let $n = 12k + a$ for some $k \in \mathbb{N}_0$, $0 \leq a \leq 11$, and let $c = \frac{a(a-7)}{12}$. For all $n \in \mathbb{N}_0$, $F_\gamma(n) = \frac{n}{3}(n-1) - 2(\lceil c \rceil - c)$.*

Proof. Let $n = 12k + a$ for some $k \in \mathbb{N}_0$ and $0 \leq a \leq 11$. Then,

$$\begin{aligned}\frac{n}{3}(n-1) &= \frac{12k+a}{3}(12k+a-1)\\ &= 48k^2 - 4(2a-1)k + \frac{a(a-1)}{3}.\end{aligned}$$

Now, we will use the Euler Characteristic formula to solve for $F_\gamma(n)$ in terms of just n and c.

$$\begin{aligned}F_\gamma(n) &= \chi_\gamma(n) - V + E\\ &= 2 - 2\gamma(K_n) - V + E\\ &= 2 - 2\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil - n + \frac{n(n-1)}{2}\\ &= 2 - 2\left\lceil \frac{(12k+(a-3))(12k+(a-4))}{12} \right\rceil - (12k+a) + \frac{(12k+a)(12k+(a-1))}{2}\\ &= 2 - 2\left(12k^2 + k(2a-7) + \left\lceil \frac{a(a-7)}{12} \right\rceil + 1\right) - (12k+a) + \left(72k^2 + k(12a-6) + \frac{a(a-1)}{2}\right)\\ &= 48k^2 + 4(2a-1)k + \frac{a(a-3)}{2} - 2\left\lceil \frac{a(a-7)}{12} \right\rceil\\ &= 48k^2 + 4(2a-1)k + \frac{a(a-1)}{3} - \frac{a(a-1)}{3} + \frac{a(a-3)}{2} - 2\left\lceil \frac{a(a-7)}{12} \right\rceil\\ &= \frac{n}{3}(n-1) - 2(\lceil c \rceil - c).\end{aligned}$$

$$= \frac{n}{3}(n-1) + \frac{a(a-7)}{6} - 2\left\lceil \frac{a(a-7)}{12} \right\rceil$$
$$= \frac{n}{3}(n-1) - 2(\lceil c \rceil - c).$$

□

$n \equiv a \pmod{12}$	$F_\gamma(n)$
$a = 4$	$\frac{n}{3}(n-1)$
$a = 5$	$\frac{n}{3}(n-1) - \frac{5}{3}$
$a = 6$	$\frac{n}{3}(n-1) - 1$
$a = 7$	$\frac{n}{3}(n-1)$
$a = 8$	$\frac{n}{3}(n-1) - \frac{2}{3}$
$a = 9$	$\frac{n}{3}(n-1) - 1$
$a = 10$	$\frac{n}{3}(n-1) - 1$
$a = 11$	$\frac{n}{3}(n-1) - \frac{2}{3}$
$a = 12$	$\frac{n}{3}(n-1)$
$a = 13$	$\frac{n}{3}(n-1) - 1$
$a = 14$	$\frac{n}{3}(n-1) - \frac{5}{3}$
$a = 15$	$\frac{n}{3}(n-1)$

Table 1. The number of faces required to produce an optimal embedding

Theorem 28. *If* $F_\gamma(n) < \frac{n}{3}(n-1)$, *then* $F_{\gamma_c}(n) < \frac{n}{3}(n-1)$.

Proof. For the sake of contradiction, suppose there existed some Cayley map embedding for K_n such that $F_\gamma(n) < \frac{n}{3}(n-1)$ and $F_{\gamma_c}(n) \geq \frac{n}{3}(n-1)$. Then,

$$\begin{aligned}
\gamma_c(K_n) &= \frac{\chi_{\gamma_c}(n) - 2}{-2} \\
&= \frac{V - E + F_{\gamma_c}(n) - 2}{-2} \\
&< \frac{V - E + F_\gamma(n) - 2}{-2} \\
&= \frac{\chi_\gamma(n) - 2}{-2} \\
&= \gamma(K_n).
\end{aligned}$$

Having $\gamma_c(K_n) < \gamma(K_n)$ is a direct contradiction to Ringel's Theorem. □

Plugging in the various a-values yields Table 1. Table 1 shows that there are 12 distinct classes that require a certain number of faces to be produced by the Cayley map embedding in order for the Cayley genus to match the genus of the complete graph. An interesting pattern to note about Table 1 is that there is a line of symmetry between $a = 9$ and $a = 10$. More formally, suppose we have two distinct a-values called a_1 and a_2, where $n_1 = 12k + a_1$ and $n_2 = 12m + a_2$. Due to this line of symmetry, we know that if $a_1 + a_2 = 19$, then $F_\gamma(n_1)$ and $F_\gamma(n_2)$ will be the same formulas in terms of n_1 and n_2 respectively. Also, Table 1 tells us that if a is even, then $F_\gamma(n)$ in terms of n is the same as $F_\gamma(n+3)$ in terms of $n+3$. Table 1 also shows that if a is odd, then $F_\gamma(n)$ in terms of n is the same as $F_\gamma(n-3)$ in terms of $n-3$. Given these observations, a couple of Tables can be made that pair off these a-values in unique ways.

$F_\gamma(n) \rightarrow$	$\frac{n}{3}(n-1)$	$\frac{n}{3}(n-1) - \frac{2}{3}$	$\frac{n}{3}(n-1) - 1$	$\frac{n}{3}(n-1) - \frac{5}{3}$
	$a = 4, 15$	$a = 8, 11$	$a = 6, 13$	$a_1 = 5, a_2 = 14$
	$a = 7, 12$		$a = 9, 10$	

Table 2. Pairs based on $F_\gamma(n)$ and the the line of symmetry

$F_\gamma(n) \rightarrow$	$\frac{n}{3}(n-1)$	$\frac{n}{3}(n-1) - \frac{2}{3}$	$\frac{n}{3}(n-1) - 1$	$\frac{n}{3}(n-1) - \frac{5}{3}$
$a \equiv 0 \pmod 3$	$a = 12, 15$		$a = 6, 9$	
$a \equiv 1 \pmod 3$	$a = 4, 7$		$a = 10, 13$	
$a \equiv 2 \pmod 3$		$a = 8, 11$		$a = 5, 14$

Table 3. Pairs based on $F_\gamma(n)$ and $n \pmod 3$

5. Disqualifying Embeddings

We can now use Table 1 to derive contradictions for Cayley map embeddings that were previously thought of as being hypothetically possible by choosing the various m_i-values and seeing whether or not such a mapping is possible. The 3 main contradictions that can be derived using Table 1 are as follows.

(1) The existence of such an embedding would contradict Ringel's Theorem.

(2) The embedding would create a non-integer number of faces.

(3) The embedding would produce a non-integer genus.

Theorem 28 is a great example of deriving a contradiction via one of the 3 options above. Theorem 28 tells us that for the majority of classes, $m_3 \neq n - 1$. That is, at least one of the elements in λ is submaximum face-generating. Therefore, an assessment needs to be done to determine an alternative, hypothetically possible Cayley map embedding that produces the most amount of faces. Notice when m_4 increases by 1 and m_3 decreases by 1, the total number of faces the embedding produces suffers a net loss of $\frac{n}{3} - \frac{n}{4} = \frac{n}{12}$ faces. When an m_5 replaces an m_3, $\frac{n}{3} - \frac{n}{5} = \frac{2n}{15}$ faces are taken away. In general, when m_i optimal face-generating elements are replaced by m_i $\frac{n}{i}$ face-generating elements, the net loss of the number of faces is $\frac{n}{3}m_i - \frac{n}{i}m_i = \frac{(i-3)n}{3i}m_i$. Therefore, the goal is to minimize $\sum_{i=4}^{\infty} \frac{(i-3)n}{3i}m_i$ for some fixed natural number n, which really means we are minimizing $\sum_{i=4}^{\infty} \frac{(i-3)}{3i}m_i$, since n is always positive. To make the notation more digestible, let $a = (a_i)$ be a sequence vector where entry $a_i = \frac{i}{3(i+3)}$, and let $m = (b_i)$ also be a sequence vector whose entry $b_i = m_{i+3}$. The goal is to determine the minimum value of the single entry in am such that m has corresponding m_i-values that can hypothetically produce a Cayley map embedding. Let the dot product of a and m be called $\sigma(m)$.

$$a = \begin{bmatrix} \frac{1}{12} & \frac{2}{15} & \frac{1}{6} & \cdots \end{bmatrix} \qquad m = \begin{bmatrix} m_4 & m_5 & m_6 & \cdots \end{bmatrix} \qquad a \bullet m = \sigma(m) = \sum_{i=4}^{\infty} \frac{(i-3)}{3i}m_i$$

The quantity $\sigma(m)$ is a calculation of the number of faces lost in an embedding divided by n compared to when all the elements of an embedding are optimal face-generating. Therefore, for any Cayley map embedding, $F = \frac{n}{3}(n-1) - \sigma(m)n$. When a value in the vector m is not specified, assume that the value is 0. Before putting these tools into practice, it is essential to create an ordering of values for m to see which embeddings produce the most amount of faces whilst not immediately producing a contradiction. There are a couple of dimensions at play. As the number of submaximum face-generating elements increase, the number of faces lost increases. Likewise, the higher the i-value of the m_i the submaximum face-generating element is assigned to, the less amount of faces that are produced. Hence, in order from least to greatest, the values of m that produce the smallest values for $\sigma(m)$ are contained in the following list.

$$m = \begin{bmatrix} 1 & 0 & 0 & \cdots \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & \cdots \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & \cdots \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & \cdots \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & \cdots \end{bmatrix}, \ldots$$

$$\sigma(m) = \frac{1}{12}, \frac{2}{15}, \frac{1}{6}, \frac{1}{6}, \frac{4}{21}, \ldots$$

Since the entries of a strictly increase, the maximum number of faces produced when there are 2 submaximum face-generating elements occurs when $m_4 = 2$, which corresponds to the fourth vector in the list above. This means if there are 2 or more submaximum face-generating elements, then the associated $\sigma(m)$-value is greater than or equal to the $\sigma(m)$-value of the first three vectors in the list. This fact assures that the first four vectors of our list truly produce the 4 smallest $\sigma(m)$-values. We can disqualify embeddings by checking to see if any of the first few m-values from the list derive any of the immediate contradictions listed towards the beginning of this section. This technique can be deployed to prove the following important Theorem.

Theorem 29. *If $n = 12k + a$ for some $k \in \mathbb{N}_0$, $a \in \{8, 9, 10, 11, 13, 14, 17, 18\}$, then $\gamma_c(K_n) > \gamma(K_n)$.*

Proof. By Table 1, $F_\gamma(n) < \frac{n}{3}(n-1)$. According to Theorem 28, $F_{\gamma_c}(n) < \frac{n}{3}(n-1)$. Ergo, there is at least 1 submaximum face-generating element in the Cayley map embedding. Hence, an assessment needs to be done on the various values of $\sigma(m)$ to see what next best embedding is hypothetically possible. We start with the next best case that produces the most amount of faces. According to the ordered list of m's, the first scenario to check is to have $m_4 = 1$. Assume such is the case. Then,

$$\begin{aligned} F_{\gamma_c}(n) &= \frac{n}{3}(n-1) - \frac{n}{12} \\ &= \frac{n}{3}(n-1) - \frac{12k+a}{12} \\ &= \frac{n}{3}(n-1) - k - \frac{a}{12}. \end{aligned}$$

Therefore, since $\frac{a}{12}$ is not an integer, the embedding would produce a non-integer number of faces, which is impossible. The next best case to assess is to have $m_5 = 1$. For this scenario, the proof will be broken up into 2 cases. If $a \in \{8, 9, 10, 11, 13, 18\}$, then

$$\begin{aligned} F_{\gamma_c}(n) &= \frac{n}{3}(n-1) - \frac{2n}{15} \\ &= \frac{n}{3}(n-1) - \frac{2(12k+a)}{15} \\ &< \frac{n}{3}(n-1) - 1 \\ &\leq F_\gamma(n) \text{ due to Table 1.} \end{aligned}$$

Lastly, assume $a \in \{14, 17\}$. Deploying a similar argument to that which was used in the previous case yields

$$\begin{aligned}
F_{\gamma_c}(n) &= \frac{n}{3}(n-1) - \frac{2n}{15} \\
&= \frac{n}{3}(n-1) - \frac{2(12k+a)}{15} \\
&< \frac{n}{3}(n-1) - \frac{5}{3} \text{ for all } k \geq 0,\ a \in \{14, 17\} \\
&= F_{\gamma}(n) \text{ due to Table 1.}
\end{aligned}$$

Thus, for both cases, $F_{\gamma_c}(n) < F_{\gamma}(n)$, and so $\gamma_c(K_n) > \gamma(K_n)$. ☐

Corollary 30. *If $\gamma_c(K_n) = \gamma(K_n)$, then $n \equiv 0, 3, 4, 7 \pmod{12}$ or $n = 5, 6$.*

Proof. This statement is the contrapositive of Theorem 29 provided that $\gamma_c(K_n) \not< \gamma(K_n)$. ☐

In fact, Scheinblum showed that for $n = 3, 4, 5, 6, 7$, $\gamma_c(K_n) = \gamma(K_n)$ [6].

Lemma 31. *For all $n \in \mathbb{N}_0$, if $F_{\gamma_c}(n) \leq F_{\gamma}(n) - k$ where $k \in \mathbb{N}_0$, then $\gamma_c(K_n) \geq \gamma(K_n) + \frac{k}{2}$.*

Proof.

$$\begin{aligned}
\gamma_c(K_n) &= \frac{\chi_{\gamma_c}(n) - 2}{-2} \\
&= \frac{V - E + F_{\gamma_c}(n) - 2}{-2} \\
&\geq \frac{V - E + F_{\gamma}(n) - k - 2}{-2} \\
&= \frac{V - E + F_{\gamma}(n) - 2}{-2} + \frac{k}{2} \\
&= \frac{\chi_{\gamma}(n) - 2}{-2} + \frac{k}{2} \\
&= \gamma(K_n) + \frac{k}{2}.
\end{aligned}$$

☐

Corollary 32. *For any $n \in \mathbb{N}_0$, if $F_{\gamma_c}(n) = F_{\gamma}(n) - k$ where $k \in \mathbb{N}_0$, then $\gamma_c(K_n) = \gamma(K_n) + \frac{k}{2}$.*

Proof. This proof is the same as the proof for Lemma 31, but instead equality holds throughout. ☐

6. Theorems for Complete Graphs with Even Order

Lemma 33. *For all $n \in \mathbb{N}$, $\sum_{i=1}^{2n-1} i \equiv n \pmod{2n}$.*

Proof.

$$\sum_{i=1}^{2n-1} i = \frac{2n(2n-1)}{2}$$

$$= 2n^2 - 2n + n$$

$$= 2n(n-1) + n$$

$$\equiv n \pmod{2n}.$$

□

Theorem 34. *If $n \equiv 4 \pmod{12}$, then $\gamma_c^{\circlearrowleft}(K_n) > \gamma(K_n)$.*

Proof. For the sake of contradiction, assume $n = 12k + 4$ for some $k \in \mathbb{N}_0$ and suppose $\gamma_c^{\circlearrowleft}(K_n) = \gamma(K_n)$. According to Table 1, $n \equiv 4 \pmod{12}$ implies $F_\gamma(n) = \frac{n}{3}(n-1)$. This means that for all $i \in \mathbb{N}$, either $|\lambda_i| = 3$ whilst $mult(\lambda_i) = 1$ or $|\lambda_i| = 1$ whilst $mult(\lambda_i) = 3$. Let t and s denote the number of λ_i's with the former and latter conditions respectively. Since λ has a total of $n-1$ elements, it must be true that

$$3t + s = n - 1$$

$$= 12k + 4 - 1$$

$$= 3(4k + 1).$$

Therefore, $s = 3(4k + 1 - t)$. Hence, s must be divisible by 3. It is commonly known that the group $\mathbb{Z}_n$ has at most 2 elements of order 3, so $0 \leq s \leq 2$. Therefore, because s must be divisible by 3 and $0 \leq s \leq 2$, $s = 0$, and $t = 4k + 1$. Since the sum of each triple is 0, the sum of all the triples must be 0. This would imply that the sum of all the elements of $\mathbb{Z}_n$ is 0, which is a direct contradiction to Lemma 33. Since every cyclic group is isomorphic to $\mathbb{Z}_n$, we know that no cyclic group can be used to embed K_n optimally if $n \equiv 4 \pmod{12}$. □

Lemma 35. *If $n = 12k + 6$ for some $k \in \mathbb{N}_0$. then 15 divides n if and only if $k \equiv 2 \pmod{5}$.*

Proof. Let $k = 5m + r$ for some $m \in \mathbb{N}_0$ and $0 \leq r \leq 4$. Then,

$$\begin{aligned}
\frac{n}{15} &= \frac{12k+6}{15} \\
&= \frac{12(5m+r)+6}{15} \\
&= \frac{60m+12r+6}{15} \\
&= 4m + \frac{6(2r+1)}{15} \\
&= 4m + \frac{3(2r+1)}{5}.
\end{aligned}$$

Since 3 and 5 are relatively prime, 5 divides $3(2r+1)$ if and only if 5 divides $2r+1$, which occurs precisely when $r = 2$. Hence, $k \equiv 2 \pmod 5$. □

Theorem 36. *If $n = 12k + 6$ for some $k \in \mathbb{N}_0$, then $\gamma_c(K_n) \geq \gamma(K_n) + k$.*

Proof. According to Table 1, $F_\gamma(n) = \frac{n}{3}(n-1) - 1$, so $F_{\gamma_c}(n) < \frac{n}{3}(n-1)$ due to Theorem 28. Hence, the next best scenario is to have $m_4 = 1$. Assume such is the case. Then,

$$\begin{aligned}
F_{\gamma_c}(n) &= \frac{n}{3}(n-1) - \frac{n}{12} \\
&= \frac{12k+6}{3}(12k+6-1) - \frac{12k+6}{12} \\
&= (4k+2)(12k+5) - k - \frac{1}{2}.
\end{aligned}$$

Ergo, a non-integer number of faces is produced, which cannot occur. Now, for the $m_5 = 1$ case, it follows that $F_{\gamma_c}(n) = \frac{n}{3}(n-1) - \frac{2n}{15}$. The question is whether or not $\frac{2n}{15}$ can ever produce a whole number amount of faces. That is, when does 15 divide $2n$? Well, since 2 and 15 are relatively prime 15 divides $2n$ if and only if 15 divides n. Due to Lemma 35, we know that 15 divides n if and only if $k = 5m + 2$ for some $m \in \mathbb{N}_0$. Accordingly, assume $k = 5m + 2$ for some $m \in \mathbb{N}_0$. Then,

$$\begin{aligned}
F_{\gamma_c}(n) &= \frac{n}{3}(n-1) - \frac{2n}{15} \\
&= \frac{n}{3}(n-1) - \frac{2(12k+6)}{15} \\
&= \frac{n}{3}(n-1) - \frac{2(12(5m+2)+6)}{15} \\
&= \frac{n}{3}(n-1) - 8m - 4
\end{aligned}$$

$$= \frac{n}{3}(n-1) - 1 + 1 - 8m - 4$$

$$= F_\gamma(n) + 1 - 8m - 4$$

$$= F_\gamma(n) - 8m - 3.$$

Therefore, $\gamma_c(K_n) = \gamma(K_n) + 4m + \frac{3}{2}$ due to Corollary 32. Thus, this results in a non-integer genus, which is again a contradiction. Next, assume $m_4 = 2$ or $m_6 = 1$. This implies

$$\begin{aligned} F_\gamma(n) - F_{\gamma_c}(n) &= \frac{n}{3}(n-1) - 1 - \left(\frac{n}{3}(n-1) - \frac{n}{6}\right) \\ &= \frac{n-6}{6} \\ &= 2k. \end{aligned}$$

Therefore, $F_{\gamma_c}(n) = F_\gamma(n) - 2k$. According to Corollary 32, $\gamma_c(K_n) = \gamma(K_n) + k$. In all other cases, $\sigma(m) > \frac{1}{6}$, which means the embedding would produce less faces than the case that was just assessed where $m_4 = 2$ or $m_6 = 1$. Hence, $\gamma_c(K_n) > \gamma(K_n) + k$. Thus, in all cases, $\gamma_c(K_n) \geq \gamma(K_n) + k$. $\square$

7. Finding the Cayley Genus of Complete Graphs with Order Congruent to 6 Modulo 12 Using $G_{\mathcal{P}}$ Graphs

Definition 37. *The $G_{\mathcal{P}}$ graph (short for group partition graph) of a Cayley map has vertex set $V = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ where λ_i and λ_j are adjacent if and only if $x \in \lambda_i$ and $x^{-1} \in \lambda_j$ for some $x \in X$.*

Definition 38. *A backtrack occurs when an edge is traversed consecutively in opposite directions.*

Definition 39. *A non-backtracking Eulerian tour traverses each edge twice in opposite directions where backtracking is allowed if and only if the tour encounters a vertex of degree 1.*

The following Theorem is due to Henderson [3].

Theorem 40. *If λ and ρ are permutations of X, where $\rho(x) = \lambda(x^{-1})$ for all x, then ρ is cyclic if and only if the $G_{\mathcal{P}}$ graph for λ has a non-backtracking Eulerian tour.*

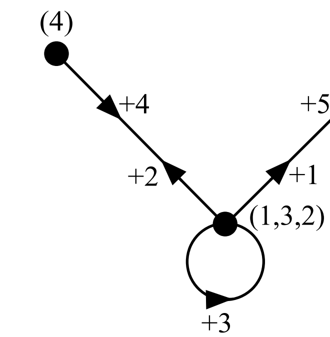

FIGURE 9. The $G_{\mathcal{P}}$ graph for $C_M(\mathbb{Z}_6, (4,1,5,3,2))$

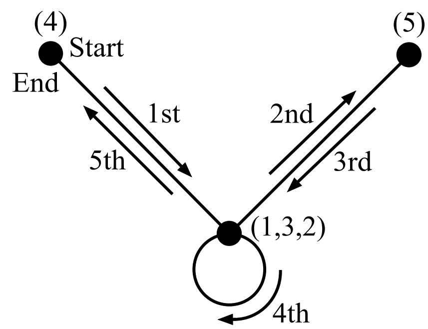

FIGURE 10. Figure 9's non-backtracking Eulerian tour

In observing Figure 9, notice that each edge's sum is congruent to 0 modulo 6 with the exception of the edge which just contains 3. This is because 3 is its own inverse under $\mathbb{Z}_6$, so this edge can be interpreted as a +3 going in both directions. When ρ is not a cycle, a Cayley map is not produced, so Theorem 40 is essential to finding λ's that actually create Cayley maps. The reason why Theorem 40 is true is because ρ is built by following the non-backtracking Eulerian path, which then corresponds to the desired λ. For example, ρ can be found by following the non-backtracking Eulerian tour demonstrated in Figure 10. The 1st edge traversed in Figure 10 corresponds to the directed edge +4 in Figure 9. The 2nd edge traversed corresponds to +1, and so on. This tour gives the order of ρ, which is $(4,1,5,3,2)$ in this case. Since the path is non-backtracking, we know that ρ will be a cycle because it will close where the tour began after traversing every other element in X. Now, to calculate λ, according to Definition 17, $\lambda(4) = \rho(4^{-1}) = \rho(2) = 4$. Hence, (4) is a factor of λ. Next, $\lambda(5) = \rho(5^{-1}) = \rho(1) = 5$, so (5) is also a factor of λ. Lastly, $\lambda(1) = \rho(1^{-1}) = \rho(5) = 3$, $\lambda(3) = \rho(3^{-1}) = \rho(3) = 2$, and $\lambda(2) = \rho(2^{-1}) = \rho(4) = 1$. Therefore, our last factor of λ is (132). Thus, for $C_M(\mathbb{Z}_6, (4,1,5,3,2))$, $\lambda = (4)(5)(132)$. Using Theorem 40, Cayley map embeddings for K_{18} and K_{30} that embed them on their Cayley genus were discovered.

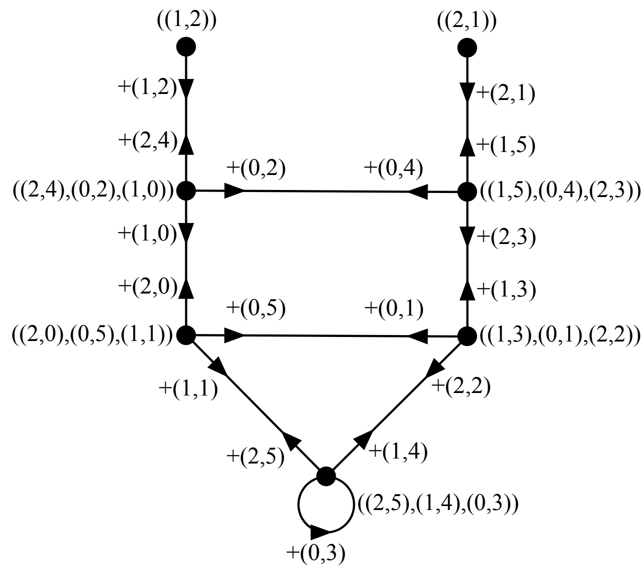

FIGURE 11. The $G_{\mathcal{P}}$ graph for $C_M(\mathbb{Z}_3 \times \mathbb{Z}_6, ((0,2),(2,3),(0,1),(1,1),(1,4),$ $(1,3),(1,5),(2,1),(0,4),(1,0),(0,5),(2,2),(0,3),(2,5),(2,0),(2,4),(1,2)))$

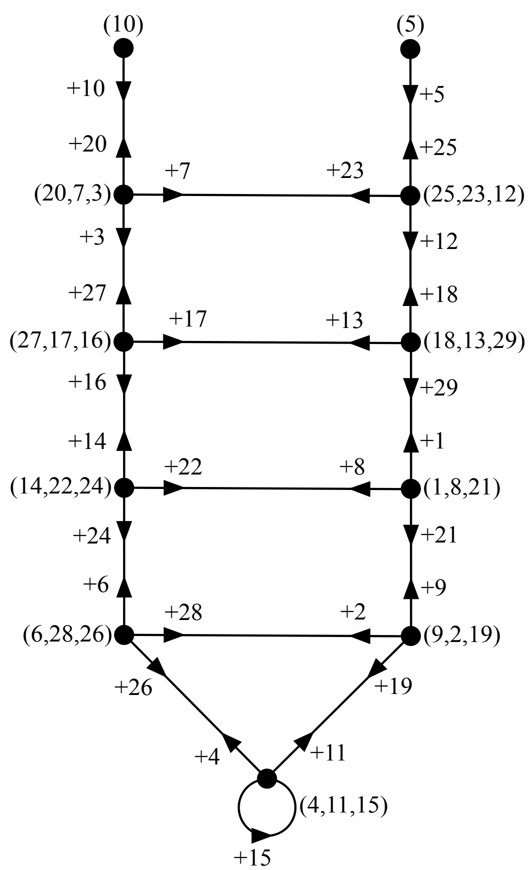

FIGURE 12. The $G_{\mathcal{P}}$ graph for $C_M(\mathbb{Z}_{30}, (10,7,25,5,12,13,16,22,21,2,$ $26,15,11,9,1,18,23,3,17,29,8,24,28,19,4,6,14,27,20))$

Figure 11's and Figure 12's non-backtracking Eulerian tours can be found by following the directions outlined in their respective ρ's. Figures 11 and 12 combined with Theorem 36 prove that $\gamma_c(K_{18}) = \gamma(K_{18}) + 1 = 19$ and $\gamma_c(K_{30}) = \gamma(K_{30}) + 2 = 61$. From Schleinblum, we also know that $\gamma_c(K_6) = \gamma(K_6)$ [6]. The calculated Cayley genera for these complete graphs provide evidence for the subsequent conjecture.

Conjecture 41. *If $n = 12k + 6$ for some $k \in \mathbb{N}_0$, then $\gamma_c(K_n) = \gamma(K_n) + k$.*

The goal is to use $G_{\mathcal{P}}$ graphs to inductively prove Conjecture 41. In regards to some observations about this class, one can see from Figures 9 - 12 that a ladder-type structure is starting to form. In each iteration, 4 vertices (or 2 rings) are being added to the ladder. In Hendrickson's paper, she proves that this type of transformation does not stop the graph from having a non-backtracking Eulerian tour [3]. This means that we can continuously increase the number of vertices by 4 whilst still having ρ be a cycle. Knowing this is useful in possibly being able to inductively prove Conjecture 41. In order to prove that the structure of a $G_{\mathcal{P}}$ graph holds inductively, there are 4 attributes that need to be preserved. Namely,

(1) the uniqueness of each element,

(2) inverse adjacency between vertices,

(3) the desired order of each factor of λ, and

(4) the ability to form a non-backtracking Eulerian tour.

In regard to some strategies that can be used to possibly inductively construct $G_{\mathcal{P}}$ graphs for the 6 (mod 12) class, it is commonly known that if $n = a \cdot b$ and a and b are relatively prime, then $\mathbb{Z}_n \cong \mathbb{Z}_a \times \mathbb{Z}_b$. This means that the group $\mathbb{Z}_6 \times \mathbb{Z}_5$ can be used to embed K_{30} onto a surface of its Cayley genus according to Figure 12. Also, we know from computer calculations that $\gamma_c^{\circlearrowleft}(K_{18}) \neq \gamma(K_{18})$. Therefore, when trying to inductively create $G_{\mathcal{P}}$ graphs for the class $n \equiv 6 \pmod{12}$, we will exclusively be working with the group $\mathbb{Z}_6 \times \mathbb{Z}_{2k+1}$ such that $k \in \mathbb{N}$ in order to optimally embed K_{12k+6}. The way this problem was attacked was to split up $\mathbb{Z}_6$ and $\mathbb{Z}_{2k+1}$ into their own separate $G_{\mathcal{P}}$ graphs, and to try to prove that GP graphs can be constructed for all $k \in \mathbb{N}$ for those two graphs separately. A structure for $\mathbb{Z}_6$ that can be used inductively has already been found and is shown in the Figure below.

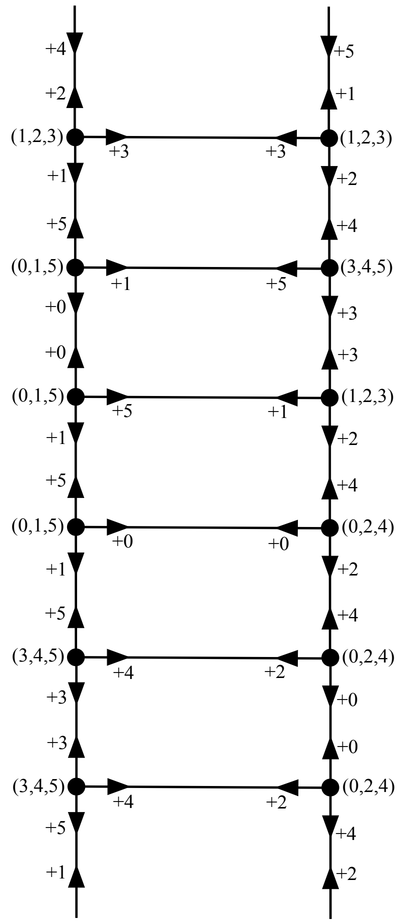

FIGURE 13. Reiterable structure for $\mathbb{Z}_6$

As can be seen in Figure 13, there are actually 12 vertices being added to the graph per each iteration for a total of 36 elements. All of the triples in the structure add up to 0 modulo 6 and all vertices are adjacent to each other's inverses in $\mathbb{Z}_6$. This ladder can be continuously reconnected to itself to produce a valid $G_{\mathcal{P}}$ graph in the $\mathbb{Z}_6$ portion with each iteration. If the same type of structure can be created for $\mathbb{Z}_{2k+1}$, and it can be proven that combining these two reiterable structures preserves uniqueness, then it would prove Conjecture 41. However, it is much harder to find a reiterable structure under $\mathbb{Z}_{2k+1}$ that preserves the desired order of each factor of λ and inverse adjacency since the group modulus is varying as k increases. With $\mathbb{Z}_6$, all the elements in the structure and before can stay the same since the modulus is constant. On the other hand, with a varying group modulus, all of the elements within the structure and all the elements before it need to be mapped to an entirely different element in order to satisfy both the element's uniqueness and having the sum of each λ_i under the modulus still be of order 1. A mapping that can create such a reiterable structure has not yet been found.

8. Finding the Cayley Genus of Complete Graphs With Order Congruent to 0 Modulo 12 Using $G_{\mathcal{P}}$ graphs

To start off with this class, we are going to try to calculate $\gamma_c(K_{12})$. According to Table 1, $F_{\gamma_c}(n) = \dfrac{n}{3}(n-1)$. Therefore, every element in λ needs to be maximum face-generating in order to produce an optimal embedding. Hence, for all $i \in \mathbb{N}$, either $|\lambda_i| = 3$ and $mult(\lambda_i) = 1$ or $|\lambda_i| = 1$ and $mult(\lambda_i) = 3$. Also, $|\lambda| = 11$. Ergo, there needs to be at least 2 λ_i's with length 1 and multiplicity 3. If such was not the case, λ would not be able to be split up into permutations of length 3 and multiplicity 1 since the number of elements left to be included in λ would not be divisible by 3.

Now that we have a better understanding of the structure of the $G_{\mathcal{P}}$ graph, we need to figure out which group would work in formulating a $G_{\mathcal{P}}$ graph to produce our ρ and λ needed to embed K_{12} onto a surface of its genus. It is commonly known in group theory that there are five groups that are isomorphic to all other groups of order 12. That is, any group of order 12 is structurally equivalent to exactly one group in the subsequent table.

Group	Order 2	Order 3	Order 4	Order 6	Order 12
$\mathbb{Z}_{12}$	1	2	2	2	4
$\mathbb{Z}_2^2 \times \mathbb{Z}_3$	3	2	0	6	0
D_6	7	2	0	2	0
$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$	1	2	6	2	0
A_4	3	8	0	0	0

Table 4. Structural properties of groups of order 12

At first glance, it seems as though any group listed in Table 4 could work in embedding K_{12} onto a surface of its genus since every group listed has at least 2 elements of order 3. In actuality, for the first four groups in Table 4, the two elements of order 3 are inverses of each other under the group operation. For example, $\mathbb{Z}_{12}$'s two elements of order 3 are 4 and 8, and $4 +_{12} 8 = 0$. The two elements of order 3 under $\mathbb{Z}_2^2 \times \mathbb{Z}_3$ are $(0,0,1)$ and $(0,0,2)$. Again, $(0,0,1)+(0,0,2) = (0,0,0)$. The following Theorem proves that λ does not create a Cayley map if there are two factors of λ that are both of length 1 that hold each other's inverses.

Theorem 42. *If there exists some $i, j \in \mathbb{N}$ such that $\lambda_i = (x)$ and $\lambda_j = (x^{-1})$, where λ_i and λ_j are factors of λ, then λ does not produce a Cayley map.*

Proof. According to Definition 17, $\rho(x) = \lambda(x^{-1}) = x^{-1}$ and $\rho(x^{-1}) = \lambda(x) = x$. Hence, (x, x^{-1}) would be a factor of ρ. Therefore, ρ would not be a cycle, and a Cayley map would not be produced. □

Now that we know that the first four groups in Table 4 cannot embed K_{12} onto its Cayley genus, the only group left that could work is the even permutation group A_4. Before we can define A_4, we must define S_4, which is the permutation group A_4 is derived from.

Definition 43. *The symmetric group of degree n (or the full symmetric group), denoted S_n, is the set of all permutations of the finite set $A = \{1, 2, \ldots, n\}$.*

In regard to calculating the order of S_n, notice that every permutation is a bijection from A to A. Therefore, for the first element of A, 1, there are n options as to where 1 can map to. There are $n-1$ elements that 2 can map to since 2 cannot map to the element that 1 maps to without contradicting the permutations bijectivity. For the same reason, 3 can only map to $n-2$ elements, and so on. Hence, $|S_n| = n!$. Typically, multiplication of permutations is performed from right to left, and we will do the same moving forward. We will look at S_4, which is of order 24.

$$S_4 = \{e, (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23), (123), (124), (132),$$
$$(134), (142), (143), (234), (243), (1234), (1243), (1324), (1342), (1423), (1432)\}$$

To demonstrate S_4's group operation, let us multiply (134) and (13)(24) together. This gives us (134)(13)(24). It is important to specify the order in which we are multiplying permutations since S_n is non-Abelian (not commutative) for all $n \geq 3$. We will first check to see where 1 maps to. Looking at our combined permutation and moving from right to left, 1 is first mapped to 3. The next time 3 shows up when moving left has it so 3 gets mapped to 4. Hence, 1 gets mapped to 4. To assess where 4 gets mapped to, 4 gets sent to 2, and there are no more 2's in the permutation when moving from right to left. Ergo, 4 simply gets mapped to 2. 2 gets mapped to 4 which maps to 1, so 2 gets mapped to 1, which closes off our first cycle (142). We do not actually need to check where 3 gets mapped to as all the other elements of $A = \{1, 2, 3, 4\}$ are accounted for, so we know that 3 gets mapped to itself. We will do so anyways to prove that such is the case. 3 gets mapped to 1, and in the left most two-cycle where 1 shows up, 1 gets mapped to 3. Therefore, 3 gets mapped to 3. This would correspond to the mapping (3), but cycles that map elements back to themselves are not included when writing out the full permutation mapping. It may be useful for the reader to try multiplying some of these permutations together to convince themselves that S_4 is closed under its group operation.

Definition 44. *A permutation is even if it can be written as the product of an even number of two-cycles.*

Every permutation in S_n, $n \geq 2$ can be written as a product of two-cycles. If $\alpha = (a_1 a_2 \ldots a_m)$ is a cycle of length m, then its two-cycle form is $\alpha = (a_1 a_m)(a_1 a_{m-1}) \cdots (a_1 a_2)$. As outlined in Definition 44, a permutation is even if it can be written as an even number of two-cycles. Similarily, a permutation is odd if it can be written as an odd number of two-cycles. The set of even and odd permutations partition the set S_n so that half of S_n's elements are even and the other half are odd when $n \geq 2$. The even permutations contained within S_n form a subgroup, denoted A_n, where $|A_n| = \dfrac{|S_n|}{2} = \dfrac{n!}{2}$ when $n \geq 2$. Notice that the set of odd permutations does not form a subgroup under S_n due to the absence of an identity element. Since $|S_4| = 24$, $|A_4| = 12$. Using A_4, a $G_{\mathcal{P}}$ graph was found that generates a ρ that optimally embeds K_{12} onto a surface of its genus.

$$A_4 = \{e, (12)(34), (13)(24), (14)(23), (123), (124), (132), (134), (142), (143), (234), (243)\}$$

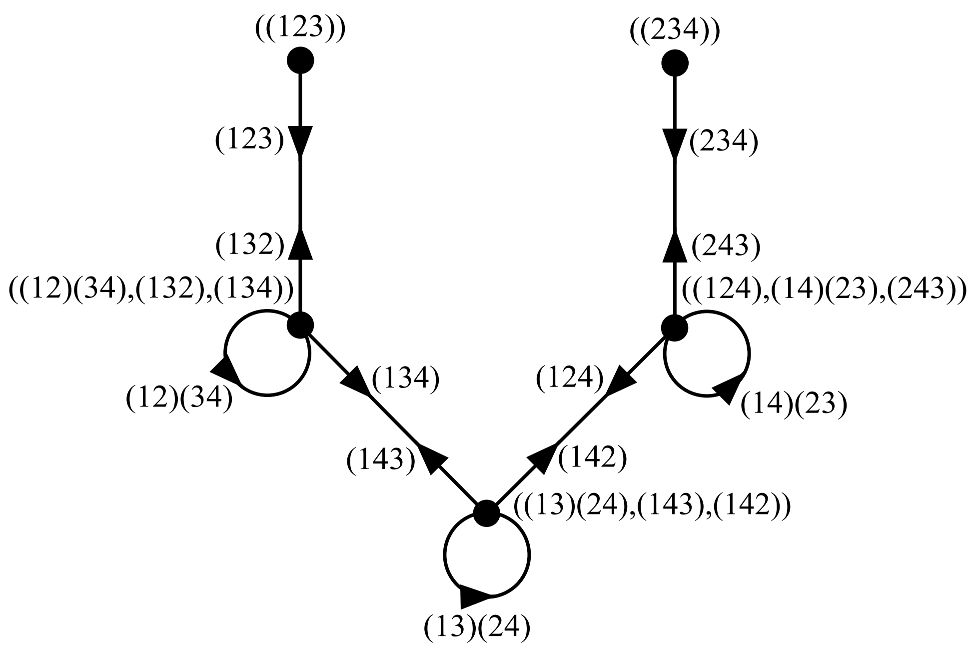

Figure 14. The $G_{\mathcal{P}}$ graph for
$C_M(A_4, ((123), (12)(34), (134), (13)(24), (142), (14)(23), (243), (234), (124), (143), (132)))$

One can notice that it is easy to find Figure 14's non-backtracking Eulerian tour, so it would be convenient to find more $G_{\mathcal{P}}$ graphs that mimic this structure of having every triple contain an element of order 2 from X. Let us analyze what the $G_{\mathcal{P}}$ graph would look like for $\mathbb{Z}_2 \times A_4$. $\mathbb{Z}_2 \times A_4$ contains 7 elements of order 2, 8 elements of order 3, and 8 elements of order 6. Hence, we have plenty of elements of order 3 to put into our two λ_i's of length 1. Also, The number of triples the $G_{\mathcal{P}}$ graph for K_{24} will have is $\dfrac{24-3}{3} = 7$, which is perfect since there are 7 elements of order 2 that can be put into each of the 7 triples. Using $\mathbb{Z}_2 \times A_4$, the $G_{\mathcal{P}}$ graph that creates a Cayley map that embeds K_{24} onto a surface of its genus was found.

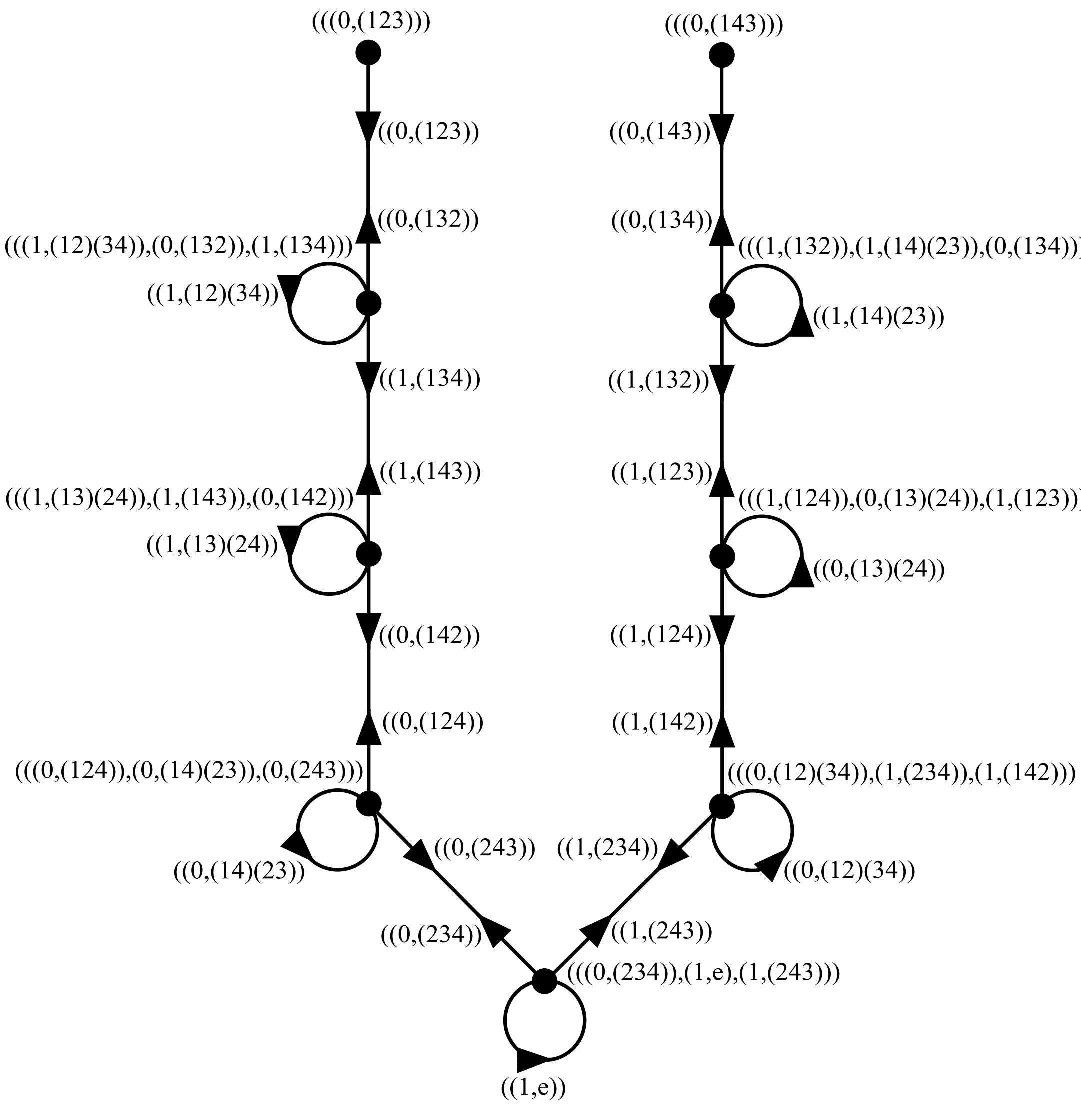

FIGURE 15. The $G_{\mathcal{P}}$ graph for $C_M(\mathbb{Z}_2 \times A_4, ((0,(123)),(1,(12)(34)),(1,(134)),(1,(13)(24)),(0,(142)),(0,(14)(23)),(0,(243)),$ $(1,e),(1,(243)),(1,(142)),(1,(123)),(0,(134)),(0,(143)),(1,(132)),(1,(124)),(1,(234)),$ $(0,(234)),(0,(124)),(1,(143)),(0,(132))))$

Figures 14 and 15 both respectively prove that $\gamma_c(K_{12}) = \gamma(K_{12})$ and $\gamma_c(K_{24}) = \gamma(K_{24})$. If you cut Figure 15 down the middle, the A_4 piece of the left side of the $G_{\mathcal{P}}$ graph was made from Figure 14. This is promising since it provides some evidence that an inductive proof to solve for the Cayley genus for a family of complete graphs may be possible by building these graphs upon one another. The previously mentioned sought after $G_{\mathcal{P}}$ graph structure that has a simple Eulerian tour occurs when deploying the group $\mathbb{Z}_2^k \times A_4$ to embed $K_{2^k \cdot 12}$. One can notice some interesting reflections about Figure 15's midline. Being able to reiterate these reflections to preserve uniqueness, inverse adjacency, and the order of each factor of λ will prove that $\gamma_c(K_n) = \gamma(K_n)$ when $n = 2^k \cdot 12$. Figures 14 and 15 provide evidence towards the following conjecture.

Conjecture 45. *If $n = 12k$ for some $k \in \mathbb{N}$, then $\gamma_c(K_n) = \gamma(K_n)$.*

9. Conclusion

This paper delves into the effectiveness of Cayley maps in embedding complete graphs of an even order. Specifically, a conjecture has been established for the Cayley genus of complete graphs with an order congruent to 6 modulo 12 and with an order congruent to 0 modulo 12, along with some speculative strategies to possibly proving them. This thesis also establishes the 12 classes of Cayley maps and proves the face formulas for each of these classes in a form that is important in the context of Cayley map embedding. Through our analysis, we note that Cayley maps cannot embed complete graphs optimally for 8 out of the 12 classes given the graph's order is greater than 6. That being said, it is still unknown as to how much worse than optimal Cayley maps are at embedding these 8 classes outside of the 6 modulo 12 class. For future research, we recommend studying one of the 12 classes, most preferrably in the 4 modulo 12 class. We know cyclic groups cannot embed this class of graphs optimally, so it would be eye-opening for more research to be done on this class. We hypothesize that the group $\mathbb{Z}_2 \times \mathbb{Z}_{6k+2}$ would embed K_{12k+4} optimally since it does so for $k = 0, 1$ [6] [3].

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