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Why Are They Called Real Numbers If They Aren't Real, and Other Such Questions?

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Introduction

Mathematics is its own kind of beast, one that presents its own unique philosophical problems. The uniqueness of mathematics as a discipline is felt by almost everyone who has had at least one formal math class: either it is denigrated as an overly abstract or useless discipline ("when will I ever need this in real life?" is the common refrain), or it is hailed as holding the true workings of the universe, a view held by many scientists and freshman math majors. The problem, then, for philosophers is how both of these positions can exist: how is that something so abstract and removed from everyday life can be so attuned to the workings of the universe? Mathematical realism seeks to answer this question in a way that has resonated with philosophers from before Socrates to the present day, a position that has been constantly reworked and reformulated to avoid arguments against opposing positions. The position that mathematical entities are real, that is, have ontological status, has been popular for centuries, and might well have been the earliest position in the philosophy of mathematics.

What it is for mathematical entities to have positive ontological status, of course, is dependent upon what sub-species of realism we are dealing with, so to speak. To "exist," then, could be interpreted as being ontologically independent of any other type of entity, or to be separable from any other entity, which is not necessarily the same thing. In addition, the existence of mathematical entities might be dependent for various philosophers on what kind of mathematical entity it is: the ontological independence of numbers or geometric objects has been more or less fashionable in different time periods.

In service of this, we ought to define what kind of a position we mean when we talk about mathematical realism. The Stanford Encyclopedia of Philosophy defines generic realism to be that *a*, *b*, and *c* exist, and have certain properties “independent of anyone’s beliefs, linguistic practices, conceptual schemes, and so on” (Miller). Hence, in any mathematical realism, mathematical objects must have their properties independent of other factors, so some kind of independence ought to be a feature of any mathematical realism. The meaning of exist is more difficult to interpret, especially when trying to apply a definition of mathematical realism to those who did not have such a term, as we attempt to do for the bulk of the first two chapters. If exist is only taken to mean mathematical propositions are true, then it would not be a very good filter: philosophers who are self-proclaimed, vehement anti-realist would be charged with saying mathematical objects exist. An easier metric is the posited relations between things: most commonly included in a philosophy being a story of causality. As we will explore later, if *x* is causally efficacious, that is, is the prerequisite for some other fact to be such a fact, *x* must exist, as nonexistence cannot be a cause. Hence, mathematical objects are principal over some other entities we accept as real. These criteria for a mathematical realist are borrowed from Zarepour in his explorations of Ibn Sina, a philosopher we talk about in Chapter 1.

The grand tradition of realism deserves a grand treatment, and cannot be easily dismissed given its hold on philosophers, generation after generation. It would be simple enough to take this position uncharitably and call it ludicrous, but ultimately disingenuous to do so. As such, I attempt to paint the broad strokes of mathematical realism and follow the historical changes the position has gone through in its most plausible forms. In service of this, I will classify what kind of mathematical entities exist for each sub-species of realism, and in what way it is they exist. Then, having the strongest arguments from the realists on the table, I will be examining the ontological implications, and formulate my response as to why realism is an untenable position.

Chapter 1: Presocratics to Pre-Enlightenment

The Presocratic period in the history of philosophy spanned the 6th and 5th centuries BCE, prior to the posited birth of Socrates, the main interlocutor in Plato's dialogues, in 470 BCE. Presocratic philosophy, seen as the beginning of Western philosophy, was particularly concerned with the "problem of the one and the many" in the realm of metaphysics, which attempts to make sense of the chaos that one experiences through sense data through organizing this barrage of the senses into what they called the *kosmos*. In trying to solve this problem, most Presocratic philosophers aimed to justify an *arche*, a first principle, which explains all of our sensory data. It is commonly claimed that the Pythagoreans, who we will be discussing shortly, had an *arche* of number, though this is a bit more nuanced than it seems on its face. The role *arche* plays in Presocratic philosophies is often described as a "material principle of everything," or described as the first principle of a philosophy (Barnes *Vol I* 9). This could be taken in multiple senses: "primary substance," "starting point or beginning," similar to calling it a first principle, or "originating cause," the last of which will be discussed shortly (Guthrie 57). Aristotle in his historical account also gives the impression that the Milesians, a subset of Presocratic philosophers from Miletus, "appeared to be concerned only with 'principles of a material kind'," referencing their material *arche* (Guthrie 82). The sense in which Barnes and Guthrie are taking *arche* is echoed in how McKirahan introduces the concept, as a "starting point, basic principle, originating source" (McKirahan 33). Even more, he goes on to refer to *arche* as that which is "capable of generating a (living) world," strengthening the idea of the *arche* as a generative force that has the most prior causal efficacy (McKirahan 36).

Hence, there are two cases: either a philosophy might have an *arche* that excludes ontological commitments to any other entities, or it may have an *arche* that is not incompatible with the reality of other entities. One example of an exclusive first principle is the atomist position, taken by Democritus in the Presocratic period. On the atomist view, similar to modern physics, all phenomena is really many tiny particles clumped together, so the only things that exist are the atoms, which represent being, and the void, that makes up non-being and is the medium in which things *can* exist. A chair, for instance, then does not exist to an atomist, but the atoms that make it up do, or at least does not exist in the same way the atoms and void does. Rather, phenomena has a secondary existence that is subsidiary to, and caused by, the interactions of these atoms within the void. Since everything that *is* is made up of these atoms, nothing is anything *but* atoms. Hence, atomists cannot ontologically commit themselves to anything aside from atoms and void, making their position an exclusive one. In the case the *arche* is non-exclusive, an ontological commitment to the first principle does not rule out an ontological commitment to anything else. For instance, a position that posits mathematics as a first principle need not exclude other objects from its ontology, unless it is contradictory to mathematical principles. However, the first principle still gives us a clue as to the ontological priority of a position, that is, what is prior to all other phenomena and causes them to act the way they do. Thales might have water as his *arche*, and this does not require him to say that seedlings are actually water, but it does require him to say that the seedling can only exist as a result of water. In fact, the water *causes* the seedling to exist, as a necessary precondition for the existence of any other object.

The use of "cause" is not accidental: causality was another important feature of ancient philosophy generally. Justifying an *arche* was not only to organize our perceptions, but also to explain the causes of things being as we perceive them: water causes things to grow, atoms group to cause us to see chairs, and so on. This need for causality was a result of the cultural context prior to Thales, specifically the reliance on mythology in order to explain why we existed, or more pragmatically, why natural disasters occurred and wreaked havoc. Hence, for a philosopher during this time period to successfully convince others of their position, they not only had to provide an organization for the barrage of sensory data, but also provide causes for seemingly senseless events. For instance, a Presocratic philosopher would not just be able to say water is their *arche*, but also argue how water as a first principle explains volcanic eruptions better than the wrath of Hephaestus. Notice, the logical relationship here is that if *x* is causally efficacious, then *x* has a positive ontological status in the given philosophy; there is no way that something could not exist and then be the efficient cause of another event.

Recall, our goal is to examine ancient philosophers and determine whether or not they are mathematical realists, beyond an explicit ontological commitment to mathematical entities. More importantly, we want to know in what way they think mathematical entities exist. So, for the reasons already stated, a mathematical realist might hold that mathematical entities hold some kind of causal efficacy, and explain the causes of phenomena we perceive. Otherwise, mathematical entities would not hold a very privileged existence, but an existence derived from other things. In other words, if mathematical entities are not causally efficacious at all, they do not have ontological priority over anything, and the entities do nothing in the philosopher's metaphysical doctrine.

In addition, since we would not want mathematical entities to have a derived existence for them to be as ontologically prior as possible, a mathematical realist ought to hold that mathematical entities have some form of ontological independence. We can understand this independence in many ways: one would be hard-pressed to find a philosopher in the ancient world holding that mathematics is purely independent, from everything. Hence, a realist might hold mathematical entities are independent of our senses, independent of our minds, or independent of the material world, most commonly. The level of ontological independence they grant to mathematical entities is what drives how much ontological priority they have: rather than being dependent upon something else, they are independent and hence the existence of other entities depends upon them, making them prior to those entities. These two factors will form our criteria for determining whether a philosopher is a mathematical realist in this chapter, since this criteria is dependent upon this historical context, and let us evaluate how strong their ontological commitments to mathematical entities are.

The earliest instance of what one might retroactively call mathematical realism in the Western tradition is the Pythagorean school, originating in present-day Italy. Since the Pythagorean school has very few surviving writings, it would be hasty to call the Pythagoreans realists without any further question, given how little we truly know about their philosophy at all. In fact, there is very little we know about Pythagoras or what he himself thought or even did, having "the wisdom to write nothing," leaving us with no evidence of his own beliefs, mathematical prowess, or philosophical acuity; in recent years there has been enough doubt cast on the idea, that we no longer believe the Pythagorean theorem originated from Pythagoras at all (Barnes *Vol I* 100). We do know that the Pythagorean school consisted of two "rival sects," namely the *mathematici*, who were concerned with the mathematical and scientific doctrine of the school, and the *acousmatici*, who kept with the religious and spiritual doctrine of the school (Barnes *Vol I* 101). Both laid claim to Pythagoras himself as belonging to their sects, but Barnes claims there is little evidence to suggest that Pythagoras was a noteworthy mathematician or scientist: but one doctrine he accepts we can certainly ascribe to Pythagoras is the immortality of the soul, a religious doctrine. As a result, Barnes claims Pythagoras to be more closely aligned with the *acousmatici*, though this does not negate the philosophical progress we are aware of that came from many who called themselves Pythagoreans, specifically *mathematici*.

The best primary sources available for Pythagorean ideas currently are from the earliest known writings from any Pythagorean, namely Philolaus, who was revived in modern scholarship on Presocratic philosophy by Carl A. Huffman in his book, *Philolaus of Croton*, where he presents fragments from Philolaus along with essays aiming to interpret his fragments. It is widely believed that Aristotle, in his *endoxa*[1] that opens the *Metaphysics*, derives his interpretation of Pythagorean ideas from the writings of Philolaus as well. As such, both Huffman and Aristotle are drawing from the writings of the same philosopher, but coming to very different conclusions on his, and by extension the Pythagoreans', metaphysical commitments. The standard interpretation of the Pythagorean school of thought is the Aristotelian one, which claims the Pythagoreans made positive ontological claims about numbers: hence the position of "numerical realism" often being ascribed to them. Aristotle claims the Pythagoreans thought that "[mathematical] principles were the principles of all things," so that we can describe the entire world in terms of mathematical entities (*Metaph*. 985b26). The world that can be described by mathematics was not just the empirical world for the Pythagoreans: it explains "nature, reason and religion as well," making it a "more universal essence" (Maziarz 12). In Aristotle's view of Philolaus's writings, "numbers seemed to be the first things in the whole of nature," which is why in the context of Presocratic philosophers,

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[1] *Endoxa* refers to Aristotle's dialectical method, in which he starts off his philosophical work by referring to the thoughts of philosophers prior to himself, and makes the historical context of his own philosophy clear. The *endoxa* acts as a historical survey for us, as well as a literature review for Aristotle, allowing him to place his own positions as preferable to those of philosophers before him.

Pythagoreans are often seen as having number as their *arche* (*Metaphysics* 968a1). From this and Aristotle's other references to the Pythagorean school, it would be hard to say that they did not have any ontological commitments to mathematical entities.

However, Huffman claims the opposite of Aristotle's position, based on what we might assume is the same evidence, namely Philolaus's writings. There are some assumptions we are accepting in this case: that Philolaus's philosophy of mathematics was typical of the early Pythagorean school and that Aristotle was basing some his view of the Pythagorean school on Philolaus's writings, alongside his interaction with the many Pythagoreans in Plato's Academy during his time there. In any case, Huffman interprets Philolaus as having number be an epistemological aid, rather than an ontological reality, that Philolaus "identifies number as a basis of the knowledge of reality" (Huffman 55). The common refrain "all is number" is a misnomer that Huffman attributes entirely to Aristotle, not Philolaus: more accurately, one could say Philolaus thought we can only know through number.

Huffman believes that Aristotle reached his conclusion, not on the basis of positive claims Philolaus made, but rather on the basis of how he carried out his philosophy. Rather than calling them mathematical realists on the basis of explicitly stated metaphysical commitments, Aristotle might attribute mathematical realism to the Pythagoreans (assuming, of course, he has been reanimated and reacquainted with the slew of new terminology that plagues contemporary philosophy) because they treat all that is real as though it is a mathematical entity. In other words, they apply mathematical theories to all that is real and use these theories to draw conclusions about non-mathematical entities. One example Aristotle gives of this is the positing of a tenth planet, the counter earth so that there would be ten, a much better number than nine, heavenly bodies in the sky, even as the number of visible bodies that Aristotle delineates in *De*

Caelo are only nine (Aristotle *Metaphysics 986a8*). There are many other apocryphal stories attributed to the Pythagoreans; throwing a man overboard for asserting the "existence," for lack of a better word, of irrational numbers, for example; all to demonstrate that the Pythagoreans took their mathematical principles as serious matters.

Huffman is correct in noting that Aristotle had the motivation to construe Pythagoreans as mathematical realists, in order to create a dialectic between them and Plato. Since Aristotle was writing his *endoxa* as a review of the historical context prior to laying out his position, largely to respond to these previous positions, he would be motivated to paint them in a more charitable light towards himself. He would also be motivated to group them in such a way as to make the philosophical discussion clearer, potentially leading to less nuanced presentations of prior positions in the *endoxa*. It is true that all these factors might be reasons not to prefer the Aristotelian interpretation of Philolaus's fragments. However, this does not fix all of the other inconsistencies that Huffman falls into by interpreting Philolaus as using number as epistemology rather than ontology.

First, we would have to reasonably assess how charitable Aristotle might be to the Presocratic position. Even as it might be convenient to frame the Pythagoreans in dialectic with Plato, it would not be necessarily helpful to concoct a specific example of counter earth. If he is faithful in his exposition of this theory, then, it would follow from this alone that he is correct in the idea that Pythagoreans are mathematical realists. If their number theory is in conflict with the sensible data they garner from the world, in that their theory says there ought to be ten planets, but they only observe nine planets, they could either give ontological priority to their number theory, or the sensible data. By positing a counter earth, then, they give ontological priority to number and therefore, must be mathematical realists to some extent. The fact of there being a tenth planet is derived from the number theoretical fact that ten is the proper number for a collection of things, on the Pythagorean view. In addition, Aristotle had access to a much broader view of the Pythagorean school as a whole, having been contemporaries with many of them. In fact, it is even conceivable he had access to more of Philolaus's writings than we, and Huffman, currently do; though, it is just as possible that some of the fragments Huffman is basing his interpretation upon were not known to Aristotle.

The question of sensible data brings us to the second objection against Huffman's argument. According to Huffman, Philolaus was an epistemic skeptic as well, believing that "the ultimate basis of knowledge is beyond our grasp" (Huffman 65). This does not address the role of number in the basis of knowledge, or how number can act as an epistemic aid at all under such skepticism. If, indeed, the Pythagoreans did not think one could grasp "true" knowledge, and that numbers are simply imitations that we can use, by proxy, to understand that which has ontological reality, then it would not make sense to put as much emphasis on number as they did. Moreover, if numbers were truly epistemic aids for the Pythagoreans, and they had deductive certainty enough to prefer numerical evidence to sensory data, this would be in contradiction with the true basis of knowledge being closed off to us. With epistemic aids as certain and with as much predictive power as numbers did for the Pythagoreans, it seems ludicrous to claim that the true basis of knowledge would be closed off to us. If numbers were perfect, and were epistemic aids, or epistemology ought to be perfect as well for being based upon these perfect objects. Hence, mathematical entities could not be purely epistemic aids for the Pythagoreans without contradicting their epistemic skepticism. Huffman does not explain this tension between ideas, and hence this stands as an argument agains his interpretation.

In addition, as P.M. Kingsley observes in his review of Philolaus, Huffman ignores other fragments, either refusing to attribute them to Philolaus or simply deeming them as irrelevant to Philolaus's overall philosophy, such as the identification of gods with angles or the identification of numbers with abstract concepts. Overall, Huffman's interpretation of Philolaus seems to be lacking a consistent explanation as to why Pythagoreans gave number the importance they did, nor does he essentially rule out the interpretation of Pythagoreans as mathematical realists by arguing that they used number as an epistemological basis. For these reasons, considering that the problems of Aristotle's account are only potential whereas Huffman's problems are actualized and evident, we prefer the Aristotelian account and will consider Pythagoreans, represented by Philolaus, to be mathematical realists.

There are scholars who do not believe this to be a fully faithful interpretation: for instance, though Barnes accepts that "some of Aristotle's account [of the Pythagoreans] bears a resemblance to the views expressed in the Philolaic texts… we cannot justly interpret Philolaus's texts by way of Aristotle's reports" (Barnes *Vol II* 77). I would not say, however, that this should affect whether or not we ultimately read Philolaus as a mathematical realist, anachronistically as it may be. As Barnes proceeds to lay out, Philolaus's metaphysics is one in which "physical objects are reduced to geometry," which in turn "is constructed from numbers," hinting at him interpreting Philolaus with a metaphysics that reduces everything ultimately to numbers and, as we shall see, to his basic principles of the unlimited and limiters (Barnes *Vol II* 81). Since scholars are in agreement that Philolaus certainly believes the unlimited and limiters to be real, the only question is whether we consider these to be mathematical. If so, then Philolaus ought to be interpreted as a mathematical realist. Barnes does exactly this, observing that Philolaus's "shapes or limiters are, after all, essentially numbers," committing him to interpreting Philolaus as some species of numerical realist (Barnes *Vol II* 93). Guthrie seems to hold that Philolaus can be interpreted as such as well, thinking of numbers as being ontologically prior and even the Pythagorean *arche* rather than the epistemic aids that Huffman posits they are: as in "the process of generation" numbers are derived from the limiters and unlimited, and "'things' [are generated] from numbers" (Guthrie 239). He even goes on to clarify that this is not a logical priority, as Huffman might claim (that numbers are logically prior to our understanding of anything) but rather temporally prior, based on Philolaus's cosmogony, which we shall describe more in depth. Hence, on yet another count we see that we ought to interpret Philolaus as a mathematical realist: that is, we ought to take his philosophy as giving numbers ontological weight, not just epistemological weight.

As we alluded to, the primary components of Philolaic ontology was "Nature… joined from both unlimiteds and limiters" (Philolaus DK 44B1). Philolaus goes on to say both that we cannot know anything "if all things were unlimited," and that "it is not possible for anything at all to be known… with [number]," implying that the limiters, which make the unlimiteds knowable, are identified with numbers (Philolaus DK 44B3-4). These principles are what form his cosmogony, a pre-*harmonia* universe in which the unlimited exists as something chaotic and unknowable. When it is joined with the limiters, then, it becomes knowable and this is Philolaus's sense of the *kosmos*. The *harmonia* is that which joins the unlimited and the limiters, and causes them to exist in harmony with one other; it is also the word used to describe the musical harmonies that correspond to ratios, another discovery for which the Pythagorean school is famous (Philolaus DK 44B6a). The shared word is fitting, since the sound of music, as yet unorganized in our perceptions, seems an unlimited, but is fitted within the *kosmos* by the limiters, namely the ratios (recall, as we said, the limiters are essentially numbers for Philolaus).

The unordered sensory data, the unlimited, is identified with the ratios, the limiters, and this identification is an instance of *harmonia* at work, which makes the organization of our sensory data possible at all.

Having established that the Pythagoreans can be considered realists, and what their realism entails, we ought to examine their position regarding the ontological status of mathematical entities. Numbers are primary for the Pythagoreans: these are the kind of mathematical entities they are concerned with the ontology of. The refrain "all is number," however, is much too general to tell us anything substantive about their metaphysical commitments. As demonstrated above, numbers had ontological priority and structured all of our sensory data, hence making them causally efficacious. In general, the Pythagorean thesis that the whole world is structured by number is one that makes everything ontologically dependent on number; but this does not *ipso facto* imply the ontological independence of number, our other criterion.

To answer this question, we might look at how separable numbers are from the sensible world for the Pythagoreans. The historical context of mathematics in Pythagoras's time was one where numbers were used in a material sense, "without being considered as purely rational entities," that is, they were not completely divorced from any materiality (Maziarz 12). In addition, if we are to agree with the Aristotelian interpretation of the Pythagoreans, then number is their *arche*, an originating source and first principle that was seen as explicitly material for the time period, as we observed in our discussion of scholarly consensus on how to understand *arche*. The number "two" would not be considered as an embodiment or the essence of the abstract concept of pairs, but rather would be a specific instant of a pair, relevant to the context: two sandals, two oranges, and so on. A report from Theophrastus on the philosophy of Eurytus, another known Pythagorean, claims that he would identify an empirical object with the number of pebbles needed to create its image: as such, a man is defined by the number 250, and a plant by 320. It is worth noting, however, that this does not necessarily mean Philolaus, our Pythagorean representative, held the same position. In addition, the Pythagoreans do not, in any known fragments or testimonies, posit any form of numbers as purely rational entities, and it would be absurd to if number should indeed be considered their *arche*, as the first principles of most Presocratic theories were material, in order to cause the physical phenomena gathered by our senses. As such, we can conclude that the Pythagoreans did not consider numbers to be ontologically separate from the sensible world, and so they could not be ontologically independent, as they do not exist divorced from materiality. Hence, the Pythagorean position would classify numbers as causally efficacious entities, but not ontologically independent from materiality. However, mathematical entities might be independent in some other sense. Instead we might examine if they are independent of individual human minds, which clearly seems to be the case for the Pythagoreans: if mathematical entities were dependent upon our individual minds, physical phenomena would be ultimately caused by our thoughts, which is certainly not the case from their philosophy. Hence, mathematical entities are also mind-independent, and we can class Pythagoreans as mathematical realist by our criteria.

The lasting impact of the Pythagorean tradition is felt in Western philosophy to this day. However, the most immediate and prominent descendant of their realist position is embodied in Plato's philosophy of math. One would be amiss to conduct any kind of philosophical project without recourse to Plato, especially regarding the ontology of abstract concepts, such as mathematics. Since the Pythagoreans are his intellectual precursor, it would be useful to delineate the points upon which Plato agrees with, or builds upon their philosophy of mathematics. As discussed, Philolaus's first ontological principles are the limited and the unlimiteds. This is echoed in *Philebus*, where Socrates agrees with Philebus and Protarchus, his interlocutors, on four kinds of things that exist (*Philebus* 36). The first and second kinds they agree upon mirror Philolaus's exactly. For Plato, the unlimiteds are those things that can be transformed into their opposites *ad infinitum*. For example, cold can be made warmer and warmer as far as one's heart desires, until cold transforms into hot, and may continue getting warmer without limit, in theory, and vice versa to transform hot into cold. These infinite divisions and transformations of one thing into its opposite are characteristic of the problem of the one and the many, and the source of the empirical skepticism that many Ancient philosophers held, as "[nothing] which is fixed [can] be concerned with that which has no fixedness," making objectively true claims impossible (*Philebus* 117). Plato solves the problem of perception in the *Republic*, which he introduces using the example of three fingers, by introducing the Forms, which are stable, unchanging ontological principles (*Repub.* 523c). Similarly in the *Philebus*, he solves the problem of the unlimiteds by imposing the other existing thing, the limited, upon them to create the third class of existing things, that which is a mixture of the limited and unlimited. The limited, for Plato, is mathematical entities-- more specifically, for his examples of music and carpentry, he is referring to ratios. In mixing these classes, or imposing the qualities of the limiteds class upon the unlimiteds, “[number] creates harmony and proportion among the different elements” (*Philebus* 32). The imposition of mathematical structure (the limited) upon the unlimiteds is similar to the Pythagrean approach to the problem of the one and many, another important way in which Plato's philosophy of math mirrors that of the Pythagoreans.

Without number, then, no kind of science (in the sense Plato uses the word) would be possible. Of course, it is not hard to make the claim that Plato was a staunch believer in the epistemic value of mathematics: he hails it as that which makes the soul look upwards (*Repub.* 523a) and a means of recollection of the Forms (*Meno* 85d), which is another similarity he shares with the Pythagoreans. In terms of the ontological commitments we can ascribe to Plato, from the four types of things that exist that Plato enumerates in *Philebus*, we can surely conclude that he posits the existence of mathematical entities (that which composes the class of the finite), but this does not tell us in what sense they exist, and how we might fit these entities in relation to the rest of Plato's ontology. In addition, he holds a view that repeats in Greek rationalist thought, from Parmenides to Philolaus to Plato, expressing the limits of our senses in informing our rational proceedings (*Phaedo* 65b).

This presents a point of divergence from the Pythagorean philosophy of mathematics for Plato. Rather than imposing the certainty and rationale of numerical principles upon the sensible world, as the Pythagoreans do in positing the counter earth theory, Plato draws a distinction between the visible and invisible existences, underpinning his notorious theory of the Forms (*Repub.* 507b8). The dichotomy between the visible and invisible realms of existence causes many scholars to refer to Plato as an ontological dualist, implying that there are two categories of existents for him: the particulars of the sensible world, and the Forms of the invisible world, where the former participates in the latter.

There are philosophical objections to the claim that mathematical entities could be Platonic Forms. One such objection is "the Uniqueness Problem," which was recognized by Aristotle: put simply, how is it that Two can be added to Two, when Two itself is unique, as all Forms must be (Annas 151)? From whence is it this duplicate Two arises? Plato cannot relegate the addition of two to two to the visible realm only: while in he makes a distinction between the invisible realm of the Forms and the visible realm of sensory data, the problem of adding two to two cannot be solved by either realm. When the farmer's herd is two sheep and they are gifted two more, they certainly have a grand total of four sheep, but this is not something particular to sheep, nor birds, nor stones, or any other particular, sensible entity. When Plato considers mathematical entities as Forms, he cannot add two and two, and mathematics without such an expression is hardly mathematics at all.

If we do intend to take the mathematical realists as charitably as possible, and consider their strongest positions throughout history, we then must add a third kind of existent to Plato's ontology: the intermediates. While the intermediates certainly solve the uniqueness problem, the evidence within the dialogues for intermediates in the Platonic ontology does not use them only as a solution to this specific problem (Annas 156). The clearest evidence, in my view, for Plato having intermediates is in the *Phaedo*, where he talks about those "that are not the Form but has its character whenever it exists," such as three which has the character of Odd but is not itself, the Form of Oddness (*Phaedo* 103e4). This suggests that mathematical entities have innate properties by nature, and to change the property of this number, it must change to become a different number: three can never participate in the form of the even, but we can *change* three by adding or subtracting the unit, which would produce something that participates in the Even. Three, then, is a particular that participates in the Odd but is not part of the visible (one would be hard-pressed to trip over three in the forest, for example) and is not a Form (or else we could not add or subtract the unity from it). By this logic, Aristotle is unfounded in saying that Plato only posits intermediates in mathematics: he employs the example of three, by nature, participating in the Odd immediately after in the *Phaedo*, drawing an analogy with the soul innately participating in the Living. Since the soul also is not a Form, since there are multiple of them, but they are also invisible and originate from the realm of the Forms, the most natural place to put it on Plato's ontological scale is with the intermediates. Further evidence for the intermediate nature of mathematical objects can be gleaned from the *Republic* VI, where Plato talks of the divided line, and how mathematics is distinct from the dialectic, as mathematics starts from the hypothesis and proceeds from there to its conclusions, rather than going back to the first principles (*Repub.* 510c). While this, on its face, can be interpreted as a claim about how a mathematician's practice differs from that of a metaphysician, the claim that mathematics "cannot escape or get above its hypotheses" is much stronger than a normative or descriptive claim (*Repub.* 511a5). Rather, because of the nature of mathematical objects, it is impossible for mathematicians to purify them completely because these objects exist as derivatives of these hypotheses that they need not justify, or so Plato says. The ontological hierarchy, then, for Plato places the Forms at the apex, followed by the intermediates (of which mathematical entities are an example), and finally by the visible realm. Aristotle's other objections to Plato's intermediates will be considered later on in the paper.

If we are to evaluate mathematical Platonism on the same criteria as we evaluated the Pythagorean mathematical philosophy, it happens that they reverse the principles of mathematical realism they emphasize. While mathematical entities are not separate from the sensory world for Pythagoreans, they do have causal efficacy; whereas for Plato, mathematical entities are certainly separable from the visible world, and in fact, exist on a completely separate plane, but not as causally efficacious as they are for Pythagoreans, as the domain over which they are causally efficacious is smaller than they are for Pythagoreans. However, Plato would accept that mathematics is causally efficacious in the sense of understanding: while mathematical entities do not *cause* the Forms to change in any way (of course, since the Forms are changeless), they can stimulate our souls to move up the divided line and better understand the Forms. Their causal efficacy, then, is lateral, affecting other intermediates such as souls or other mathematical entities (the evenness of two affects all of its multiples, for example) and vertical downwards (relative to the divided line), as it structures our visible world alongside the infinite Forms, and causes the empirical world to behave in such a way as to align with them.

An analysis of ancient philosophy would be incomplete without Aristotle, either using his own philosophy or using his accounts of what ancient philosophers thought. Just as Aristotle requires identifying the *differentia* of any class in order to properly define membership of that class, it would be useful to identify what makes the Pythagorean and Platonic mathematical realism different from Aristotle's account so as to best define what it means to be a mathematical realist for the ancient philosophers.

As is always necessary for an analysis of Aristotle's position, there is some groundwork to lay as far as terminology is concerned. The essential (pun unintended) components of Aristotle's metaphysics consist of the four causes and his idea of substances and categories (*Physics*). Aristotle's four causes are intended to fully describe, with the appropriate distinctions, why things are the way that we perceive them; it is a consequence of his scientific approach, which begets modern scientific inquiry, as well as his appeal to common-sense notions, which leads him to claim certain things to be evident without argument.

His four causes are best explained by way of example: consider a mug. The material cause of a thing is "that out of which a thing comes to be and which persists," and in the case of the mug its material cause is ceramic: even if the mug were to shatter, it would still be ceramic and this would be what persists (Aristotle *Physics 2.3.194b24*). Notice, that the ease of chipping only occurs when the mug is ceramic; a metal mug faces no similar difficulty. The formal cause of a thing is its definition: it would be the cause of those properties belonging to the mug *qua* mug. One could fill it with coffee and drink it *because* it has the form of a mug. The efficient cause is the physical forces that act upon it: pushing the mug will cause it to move. The final cause is that for which end the mug is: the mug is shaped in the way it is so that most people can drink from it. The latter two causes are less important for the discussion of Aristotle's ontology of mathematical entities: more important is the distinction between matter and form, and which is more primary.

Substances, the primary "stuff" of Aristotle's metaphysics, are mixtures of matter and form, though Aristotle recognizes that "form indeed is 'nature' rather than the matter," because the form lends a thing its "essential whatness," or quiddity (*Physics* 193b9). However, the form must inhere in something sensible-- they cannot have an independent reality outside of the senses. This is not contradictory with the form being the quiddity of any substance: "priority in *logos* is not equivalent to ontological priority," so even if the definition of a thing is needed prior to reasoning, or even talking in a cogent manner, about an object, it is not necessary for the abstract definition or "Form itself" to *actually exist* prior to the substance existing (Goldin 695). In addition to the form or quiddity of a thing, there can be other properties of a substance, that Aristotle discusses in the *Categories*: these involve the qualities of the substance, such as its color, shape, position, location, and so on. These, also, do not have independent or separable existence for Aristotle and seems contradictory to him to suppose so (*Metaphysics* XIV).

Recognizing, then, how mathematical entities fit into Aristotle's schema will tell us his stance on the ontology of mathematical entities. Dealing first with mathematical entities *qua* mathematical entities, these certainly are not sensible: even a Platonist would accept this point. Hence, they cannot be primary substances, which are *ipso facto* sensible. Much like Plato takes mathematical entities to have essential properties, Aristotle recognizes that numbers in themselves, for example, divorced from any particular set of objects or measurement, are definitions. However, if we are to understand them as the number of things, perhaps two *qua* the tenured professors in the Rollins Mathematics Department, then this is an accidental rather than essential property and is relegated to a category or predicate. What is worth noting is, for Aristotle, that simply because we can perceive of something as being distinct from another, it is not necessarily a distinct entity, as it does for Plato, for whom the latter are intermediates and the former are Forms. In whatever sense one takes them, as Marcus and McEvoy observe, "mathematical objects are neither in sensible objects nor separate from them" (Marcus and McEvoy 68).

The bulk of Aristotle's philosophy of math, however, can be found in the *Metaphysics* Book M, where he explicitly talks about the separability of mathematical objects from primary substances that are compounded form and matter, specifically that "it is not possible that such entities should exist separately" and "[exist] always along with the concrete thing" (Aristotle *Metaphysics 13.2*). He also corrects some misconceptions about mathematical entities, as one might suppose that mathematical entities would be prior to sensible objects, but for Aristotle mathematical objects are "prior in definition… [but not] also prior in substantiality" (Aristotle *Metaphysics 13.2*). He concludes that this must mean that if mathematical entities exist at all, they "exist in a special sense" (Aristotle *Metaphysics 13.2*).

He claims that they do, in fact, exist in this special way, since "things which are inseparable exist," having already established that mathematical objects are inseparable

(Aristotle *Metaphysics 13.3*). He also talks of how it is a mathematician practices, since though mathematical objects are not separable, they are "in thought separable from motion… nor does any falsity result if they are separated," specifically only within the mind of the mathematician (Aristotle *Physics 2.2.193b31*). The mathematician, as Aristotle describes in both the *Physics* as *Metaphysics*, involves the mathematician abstracting away all the other properties from sensible objects, including their sensibility. As Hussey suggests, this begs the question of whether they "start with the objects of mathematics already in existence" or whether this abstraction "create[s] or reveal[s] mathematical objects," setting up the classic dilemma between whether mathematical objects are discovered or invented (Hussey 116). Since Aristotle takes mathematical objects to be definitionally prior, it must be the case that they are already in existence, and it is only if they already exist can we know how to abstract them from sensible entities. However, due to this abstraction and inseparability, "the existence of mathematical objects must be bound up with the… existence of sensible objects" (Hussey 115). As a result, mathematical objects have an existence dependent upon some actual or potential sensible entities that might be abstracted into the mathematical objects, meaning they are not principle to other entities, but dependent upon primary substances for Aristotle: as we might expect, given that they are, indeed, primary.

As far as the separability of mathematical entities is concerned, Aristotle has no such doctrine and finds it absurd. The forms must always be inhered in some matter for Aristotle, and do not have an ontological status sufficient for mathematical realism outside of their instantiation in matter. Their causal efficacy is also limited: while he does have formal causes, this would only apply when considering mathematical entities *qua* mathematical entities. Hence, mathematical entities can formally cause certain properties to be true of other mathematical entities *qua* themselves: this is not causation in the sense that might lead us to posit the existence of an entity, but is because mathematical objects are definitionally prior, and hence a type of causality that can be attributed to definitions. For example, a bachelor is an unmarried man *because* this is exactly the definition of a bachelor, and this makes for an uninteresting sense of causality. Then, in the strictly ontological sense, for Aristotle, mathematical entities are neither separable nor principal for anything, but only attributes of things. It would be absurd, then, to conclude that Aristotle is a mathematical realist, and effectively demonstrates a counterpoint to mathematical realism in the ancient world.

It is at this point in the chronology of the position of mathematical realism that there is a discontinuity in the so-called Western tradition[2]. After the fall of Rome in the 5th century, the traditional West was unable to access the knowledge of their predecessors such as Aristotle, Plato, and the Pythagoreans: the intellectual seat moved to the so-called East, and the philosophical legacy of Greek classical philosophers was carried on by Islamic and Arab philosophers during the Golden Age of Islam (Rubenstein). Thus, to accurately portray the ongoing philosophical conversation that developed the realist position, from the Presocratics to the present, it is necessary to examine some of the most important philosophical voices from the Medieval period on the philosophy of mathematics from the so-called East. The main motivation of Islamic philosophers in this time period runs parallel to that of the Catholic philosophers after the rediscovery of Aristotle's works: trying to reconcile the painstaking reasoning and logic characteristic of Greek philosophy with their theological convictions. This was not at the forefront of every Medieval Islamic philosopher's mind, but rather part of the philosophical

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[2] The use of so-called is simply to draw attention to the fact that the "West" and "East" are arbitrary constructions designed to compartmentalize philosophy, usually in a way to best service those in positions of power. While Rubenstein recognizes that the "cultural chauvinist" (7) would not like to credit the so-called East with the philosophical progress the so-called West later built upon, it is worth calling into question the existence of this dichotomy in the first place.

conversation: much like looking for the *arche* and first principles was the main theme in the philosophical conversation of the Presocratics, but not necessarily the only method of doing philosophy, so was reconciling faith and logic for Muslim philosophers.

One of the most well-known Medieval Islamic philosophers is Ibn Sina[3] (980-1037), a Persian polymath who is celebrated for his commentaries on Aristotle and Galen, and his own innovations in metaphysics, the natural sciences, psychology, and medicine (amongst others). The fact that his philosophical treatises are framed as commentaries of Aristotle might lead to the misconception that Ibn Sina held all the same positions that Aristotle held. However, the purpose of commentaries in this scholarly tradition, rather than simply providing an exegesis of an ancient text, also involved "challeng[ing] it with criticism, and attempt[ing] to resolve these critical charges," which led to more extensive philosophical position (Ighbariah and Wagner 3).

Retrospectively, it is useful to treat Ibn Sina and Aristotle separately, due to the difference in how they are perceived in modern scholarship: while we have already argued Aristotle is not a mathematical realist, there is much more contention on whether to consider Ibn Sina a mathematical realist or not. In light of their post-Aristotelian philosophy, complete with the new terms and distinctions available, the question of whether mathematical objects exist can be more accurately, for our purposes, rephrased as whether "mathematical objects [are] mind-independent substances" (Zarepour 5). On Zarepour's account, Ibn Sina accepts that mathematical objects are substances, but are not mind-independent, even if they are not essentially mental constructs. This allows them some kind of reality, but not divorced entirely from the mental realm, and only sometimes divorced from materiality. Ibn Sina's definition of the "mathematical sciences" are those that "treat the consideration of *existents* inasmuch as they separate from materials of

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[3] His name is commonly Westernized/bastardized as "Avicenna."

particular species in *cognitive* apprehension, but not in subsistence,"[4] placing them as an intermediate between natural philosophy (which can never be separated from materiality, even cognitively) and metaphysics (which is always separated from materiality and is completely abstracted) (Avicenna *Isagoge 1.2*). Hence, Ibn Sina draws a distinction between ontological dependency on materiality for mathematical objects, and epistemic, or in his terminology, cognitive dependency. Since mathematical objects can exist in either the mental or extramental realm, and in the former they are not independent from the mind, they are therefore not separable from materiality in the ontological sense.

While Aristotle's position lumps both geometrical and arithmetic objects in one category, both of which are mixed with materiality by necessity, Ibn Sina draws a distinction between geometric (continuous) objects and arithmetic (discrete) objects, in order to more carefully assess their cognitive dependence on materiality. Geometric figures cannot be separated from materiality in any case, so that "the concept of an immaterial triangle is self-contradictory and unintelligible" (Zarepour 9). Number, on the other hand, can be separated from materiality by the estimative faculty[5], as they "[involve] a degree of abstraction that does not require the specifying of matters of certain species" (Avicenna *Isagoge 1.2*). This does not put numbers in the realm of metaphysics. Numbers can be apprehended as an admixture with matter (in understanding a bag of seventeen oranges, for example), or without, whereas metaphysical objects cannot be understood in an admixture with materiality (a specific of a politician being charged with fraud tells us nothing about the metaphysical object of justice, for example, and would constitute a moralistic fallacy). Numbers, then, are contingently mixed with materiality, whereas geometrical

[4] Emphasis mine.
[5] The estimative faculty (*wahm*) plays an important part in Islamic philosophy of mind; this is the faculty that is in charge of organizing the chaos of the Cosmos, as the Greeks would put it, and takes charge of abstraction. In other words, the estimative faculty is what populates the mental realm.

objects are necessarily mixed with materiality (though not any species of matter, as with the objects of natural philosophy[6]). This is sufficient to conclude that Ibn Sina does not allow for the separability or independence of mathematical objects, but allows them substance separable from *specific* matter in the mental realm.

Zarepour also pushes forth an interpretation of Ibn Sina as a literalist, which posits that mathematical objects *literally* exist in the extramental realm: not only do objects in the extramental realm imitate mathematical objects, but a perfect circle, insofar as it's a mathematical object, can be found in the empirical world. His reading of Ibn Sina as a literalist is a consequence of his emphasis that mathematical objects are not separable from materiality, and extracted from the sensible world by the estimative faculty (Zarepour 20). If it can be extracted, it follows that it actually exists within the specific matter it is extracted from. Hence, mathematical objects must have extramental existence-- but, again, this is not independent, but rather as an accidental property to sensible bodies, following the Aristotelian conception of mathematical objects as properties of bodies, rather than as substances in themselves. As such, they have causal efficacy in the same sense that they do for Aristotle in the extramental world, and so even on a literalist reading, Ibn Sina does not satisfy our criteria to be called a realist-- but provides a more nuanced understanding of the Aristotelian view, in which mathematical objects are real, but not substances *qua* themselves.

However, even though Ibn Sina is one of the most well-regarded Islamic philosophers of the Medieval period, his philosophy of mathematics was not the only position taken. Al-Biruni, for example, a contemporary of Ibn Sina and a polymath in his own right, has a position much

[6] It is worth noting that this distinction may not hold for Newtonian physics, for example, in which the motion of bodies is studied divorced from the species of matter it is made up of: whether the ball being thrown is rubber or plastic does not matter to the Newtonian physicist.

more reminiscent of Plato and Pythagoras. While Plato has a theory usually characterized as ontological dualism (explicitly, at least, as discussed with regards to the intermediates), al-Biruni explicitly posits two levels of perception, which map onto Plato's realms of Being and non-Being, and *three* levels of existence. Al-Biruni actually distinguishes the two levels of perception in his anthropological work, *India*, to explain the differences in belief about those things that he thought to have "transcendental meaning," such as mathematics (Samian 149). A contemporary thought experiment might do well to delineate al-Biruni's perceptual dualism: consider aliens that have discovered and landed on Earth, and share the ins and outs of their spaceship with Earth's best engineers. At the first level of perception, the mathematics that the aliens would have used to construct their spaceship would be just that to Earth's engineers: alien. Their notation would, most probably, be completely foreign, and the way in which they apply mathematical concepts or solve certain problems, even after translating their notation, might seem counter-intuitive to Earth's engineers. At the second level of perception, however, after the Earthlings and aliens talked through the mathematics on the page and reconciled it with the human picture of mathematics (possibly with some mathematicians in the room alongside the engineers), both would be able to see what the mathematics *really* means, behind the cultural constructions and usages of the mathematical entities, at the abstract level.

This seems like it ought to be uncontroversial to a mathematical realist; why, then, would al-Biruni posit a third level of existence? This is how he reconciles his philosophy of mathematics with his theological and metaphysical convictions. Much as the ancients had their most primary ontological principle-- number, for the Pythagoreans; the Forms, for Plato; and substance for Aristotle-- the most primary ontological given for al-Biruni, and other Islamic philosophers, is God, and God's reality is the truest, or highest, level of reality. As such, the three levels of existence for al-Biruni are the physical, the mental, and the metaphysical. The physical existence of mathematical objects mirrors Ibn Sina's position, which is that physical objects have mathematical properties-- for example, in flowers, "the number of their leaves… is in most cases conformable to geometry," causing "the number of their leaves to always be 3 or 4 or 6 or 18" (al-Biruni *Chronology* 294). The mental constructions of mathematical objects, even though they transcend the physical existence of mathematical objects, are still "reflections of something real," because the *real* mathematical objects are independent of any specific mathematician's mind: two plus two is four regardless of whether any specific person is holding that mathematical truth in their mind at a particular time (Samian 150). Then, aside from their physical and mental manifestations, mathematical objects have metaphysical existence, their existence "in reality"-- for example, while in the physical or mental level of existence, one can be infinitely divisible, "'One' (*al-wahid*) is in reality indivisible" (al-Biruni *Elements of Astrology* 24). One actually ends up not being a number for al-Biruni, which interestingly holds true for the Pythagoreans as well, since, in the semantic sense, he defines number as a sum of units. In the metaphysical sense, however, One is not a number because it is much more than that-- it is one of the manifestations of God. Samian makes an argument for al-Biruni drawing an analogy between the Creator and Their creations with one and the natural numbers, as both natural numbers and God's creations "can be derived from one without the one ever losing its original identity in the sense of 'losing' part of itself" (Samian 151).

On the question of whether mathematical objects are mind-independent substances, al-Biruni would certainly agree they are mind-independent, but not necessarily that they are substances. While mathematical objects are transcendental, and exist independently of any single mind, that does not make them independent in every sense for al-Biruni. Since his highest ontological principle is God, the existence of everything, mathematical objects included, is maintained by and dependent upon God. Hence, the existence of mathematical objects is “never Absolute Existence because they are continuously existing and perishing,” and mathematical objects cannot be substances in the Aristotelian sense (Samian 155). Since al-Biruni is not an Aristotelian, however, applying Aristotle’s criteria for a substance to his philosophy is inappropriate, and I believe that his philosophy of mathematics, in which mathematical objects have existence on all three levels of reality, is sufficient to consider al-Biruni a mathematical realist. In addition, while al-Biruni’s philosophy has some Platonic elements to it, especially in his perceptual dualism and recognizing the metaphysical existence as reality, he actually solves the problems of the intermediates by explicitly stating them and how they compare to the other two levels. For these reasons, al-Biruni’s philosophy of mathematics is an important step forward in the chronology of mathematical realists.

The rest of the story continues after the ancient texts are restored to the so-called Western world through the translations in Toledo, Spain. From the “beginning of philosophy” in Miletus with Thales, up until the Middle Ages, mathematical realism was an influential theory, leading to major schools of thought and present in the writings of many important philosophers. The philosophy of mathematics, of course, must be taken in the context of the mathematics available to the philosophers during this time period, the extent of which was largely to do with arithmetic and geometry-- much of the sophisticated mathematics that underpins most of the modern world, such as calculus, was far removed from the conceptual spheres of these philosophers. This is to say, the positions they held were a product of their epistemic standing-- it would be inappropriate to try and refute Plato’s theory of the Forms, for example, with 20th century mathematics. The fact that such refutations can be made, though, is what drives progress in the philosophy of mathematics, and leads to revisions of mathematical realism, all the way up until contemporary times. To dismiss or reduce mathematical realism to Plato, or Pythagoras, or al-Biruni, would be a disservice to the serious philosophical work made by those standing upon their shoulders to revise the position in light of mathematical progress.

Chapter 2: Scientific Revolution to Present

After the translation of ancient texts in Toledo in the 12th century, Europe was back on the track of scientific and philosophical progress. There was a feedback loop between the scientific development and philosophy, especially metaphysics, that led up to and powered the Enlightenment era. When Nicolaus Copernicus, a renowned mathematician, astronomer, and Catholic canon, published his seminal work on astronomy, *On the Revolutions of the Celestial Spheres* in 1543, this triggered what many historians regard to be the start of the Scientific revolution, and the shift away from humanist philosophy. The Copernican model of the universe no longer had a stationary Earth at its center, as most models before him did. Instead, the Earth, and humans, by extension, were simply one piece in the cosmological puzzle, given no special privilege by the laws of Nature that scientists in the 16th century and later were uncovering. The Scientific Revolution also saw discourse on what constitutes science, forcing philosophers and scientists (for most scholars at the time were both) to reckon with the weight to be given to reason versus evidence: this discourse is often framed in textbooks on modern philosophy as the dichotomy of the rationalists and the empiricists. The use of purely theoretical thought experiments and chains of logical reasoning to come to conclusions about the sensible world, rather than practical experiments, was a fairly uncommon methodological practice when used by Galileo in his *Dialogue on the Two World Systems*, published in 1632. For example, in this work, Galileo draws an analogy of the moving Earth with a moving ship, in service of a counter-argument to the Aristotelian argument for an unmoving Earth. By the Aristotelian argument, cited and held as the most dominant position in the Catholic Church at the time, if one were to drop a stone from the mast of a moving ship, the stone would hit the deck further from the base of the ship, in the direction of the ship's motion. Since this does not happen when a stone is dropped from a tower on Earth, then the Earth cannot be moving. The first assertion, that the stone would fall away from the base of the mast, is what Galileo sought to disprove. Rather than conducting an experiment and recording his results, he uses a chain of deductions, as "it is necessary it should happen this way," that the stone would fall at the base of the mast of a moving ship (Galileo "Dialogue" 164). Galileo's claim of necessity, a bold claim of deductive certainty about the sensory world that the ancients had so little faith in, set the tone for the scientists and philosophers of this era.

The use and subsequent adoption of Galileo's rational methodology marked the beginning of the mechanistic philosophy that characterized Enlightenment era philosophy, and was heavily championed by René Descartes, the French philosopher and mathematician, and his followers. In the view of mechanistic philosophy, one can think of the universe as a clock, such that the actions of all bodies in the universe, including humans, follow certain immutable laws, the laws of Nature itself, and this is necessary to maintain the order of the universe. In other words, humans are just bodies in motion. To discover the workings of the universe, then, one must uncover the laws of Nature that cause it to work *by necessity*. The first principles that philosophers of the Enlightenment era were concerned with were quite different from those of the ancient philosophers we saw in the last chapter. While the ancient philosophers were generally looking for first principles in a very material sense, the most primary *substance* that transforms or underpins everything in the sensible world, and causes it to have certain properties, modern philosophers were looking for first principles as rational laws that everything adhered to, regardless of what kind of material it was made of. To put it simply, modern philosophers were concerned with metaphysical first principles *entirely divorced* from materiality. To that end, then, it would be inappropriate to use the Aristotelian language of substances when talking about the metaphysical stances of modern philosophers; by and large, these people did not care for primary substances, but wanted primary principles.

We have the context of philosophy in this time period, but more specifically, this informed the kind of conversations about mathematics in philosophy. While the moderns have a vastly different context they are philosophizing in, they do still share with the ancients an admiration for Euclid's *Elements*, which pops up as an example of the gold standard of logical reasoning. For these reasons, mathematics continues to hold a special position in modern philosophy, for rationalists and empiricists alike, as a set of undoubtedly true propositions. Then, to identify the mathematical realists in this time period, we must identify not what they hold to be true, but what they hold to be a necessary law of Nature. In other words, we must identify which philosophers argue that "[the universe] is written in the language of mathematics" (Galileo, "Assayer" 238).

To operationalize this criterion, there were two main questions surrounding mathematics that philosophers of this period were attempting to answer: the age-old question of the unreasonable effectiveness of mathematics, and the universality of mathematics. Any comprehensive philosophy of mathematics for the moderns had to answer how it was that mathematics could be applied down to the real world, but also how it was mathematics could be abstracted up into a general discipline that could apply to more than one real world situation. This corresponds to the two possible sources of mathematical cognition. If the essential property of mathematics is how effective it is at predicting the future of real-world phenomena, it would make sense to draw mathematical cognition from the empirical world, much as a science does. However, if the most essential feature of mathematics is its universality, both to real-world phenomena and purely intellectual arguments, then one might find it wise to ascribe mathematical cognition to innate ideas. Shabel calls these the applicability demand and apriority demand, respectively: how a philosopher meets these demands tells us how far their ontological commitments for mathematical objects and propositions go. Hence, to judge a modern philosopher to be a mathematical realist, we draw (an imperfect) analogy with the criteria for an ancient philosopher to be a mathematical realist. While an ancient realist requires mathematical entities to be separable from the empirical world, a modern realist requires it to be applicable to the empirical world, but not dependent upon it. While an ancient requires mathematical entities to be mind-independent, a modern requires that they are *a priori* and existent/true regardless of a singular person thinking of them.

René Descartes, one of the most prominent figures of the Scientific Revolution, is well known for his work in mathematics, such as the Cartesian plane and analytic geometry, as well as his work in the field of optics. As far as his philosophical works are concerned, his most well-known work is the *Meditations on First Philosophy*, in which Descartes attempts to create a stable foundation for all knowledge, an inherently epistemological, rather than ontological, project. Since the question of mathematical realism is an ontological one, there is no direct answer to the question of Descartes's realist leanings, in spite of the frequent appeals to the truth of mathematical propositions. In light of this uncertainty, I will argue that Descartes is not a mathematical realist, and then argue the contrary position, after which Descartes's position can be more fully evaluated.

Within the *Mediations* itself, while Descartes repeatedly refers to the obvious truth of mathematical propositions, such as "two plus three is five" and "the sum of angles in a triangle is two right angles," he also always qualifies this with the fact that the truth of these propositions is not dependent on the existence of these mathematical objects. The arithmetician or geometer can produce many true statements "indifferent as to whether these things exist" (Descartes 61). While this by itself is not damning to say that Descartes holds that mathematical objects do not exist, he also holds that mathematics is not a necessary consequence, either of human consciousness (being a "thinking thing") or of sensory experience (the corporeal world), since "there is no necessity for [him] to ever imagine a triangle" (Descartes 90). This clashes with the position of early mathematical realists from Chapter 1, where the development of mathematics and the conception of mathematical objects are necessary to make sense of the world: both to impose structure on the corporeal world and make predictions, such as the Pythagoreans using ratios in understanding musical notes, or to resolve contradictions in the corporeal world, such as Plato wrestling with how one finger might be both longer and shorter, relative to the middle finger or the thumb. These are all empirically driven reasons to accept the necessity of mathematical objects, however Descartes is attempting to build a philosophy from first principles that are assuredly true: he only allows the conclusions he can reach *a priori* to be true, because "the senses are sometimes deceptive" (Descartes 60). Perhaps Descartes's reasons for not clearly positing the existence of mathematical objects are only a consequence of the type of philosophy he is engaging in with the *Meditations*, namely a strictly deductive, analytic, epistemological treatise.

For further clarification, we can look towards Descartes's other works, to see what can be gleaned from his scientific and mathematical treatises on the existence of mathematical objects. On this basis, many Cartesian philosophers make the argument that Descartes is actually a precursor to anti-mathematicism, a position that gained prominence in the eighteenth century, which holds that there are limits to the usefulness of mathematics in describing the corporeal world, as a counter to Galileo's position of a universe written in the language of mathematics. On this interpretation of Descartes, he "does not value mathematics or rigor for its own sake," relegating the status of pure mathematics to training for using deductive reasoning in the sciences (Nelson 3499). Even in his work in mathematics, laying the foundations for analytic geometry, it is clear that Descartes thought "solving mathematical problems [had] only instrumental value," focusing solely on analytical proofs that directed one towards an answer of a problem and not synthetic proofs, which were directed towards epistemic advance (Nelson 3488).

Moreover, he raises problems with the application of mathematics to the physical world, one of which can be described as "asymptotic matching," which highlights the descriptive limitations of mathematics (Wilson 3467). For example, Newtonian classical mechanics was being developed during Descartes's time, including the concept of conservation of momentum. There are certain assumptions made in order to describe the conservation of momentum mathematically, such as that the two bodies colliding with one another are rigid and inelastic, so they do not deform at the moment of impact and all of the kinetic energy involved in the collision remains as kinetic energy. With this assumption, the conservation of momentum is pieced together by looking at momentum prior to collision and after collision, but not at the point of collision, making this model discontinuous. Piecing together these two pictures without modeling what happens between the two pictures constitutes "asymptotic matching," and these unfounded assumptions and discontinuous models are among the objections Descartes raises with applied mathematics. These objections are similar to modern anti-mathematicism in discussions surrounding the sciences, but also disciplines such as economics, where assumptions about the rationality of all actors lead to theories severely lacking in descriptive or predictive power. In comparison to the criteria for a mathematical realist used in Chapter 1, Descartes would fail to meet the causal efficacy criterion, since the truth of mathematical propositions does not translate to the truth of propositions about physical or corporeal entities. In other words, the universe is not written in the language of mathematics, as far as Descartes is concerned; a fact further demonstrated by the lack of mathematical justification or calculation in his works such as *The Principles of Philosophy* (Nelson 3492).

On the other hand, there may be evidence to suggest Descartes is a mathematical realist, which is what one might expect from a rationalist philosopher following Galileo. One might expect this from a rationalist because of the primacy placed on *a priori* reasoning over empirical data; since mathematical propositions fall under the purview of the *a priori*, or at least are largely perceived to during this era of philosophy, this would lend primacy to mathematics as a discipline over any other, doubly so due to the certainty of its deductive arguments that is lacking in any empirical work. More specifically, for Descartes, "the ultimate principles" are "[t]hought, extension, and God" (Nelson 3497). In the *Meditations*, however, the first thing that he can be sure of, on which he builds his certainty for the existence of the other two principles, is that he is "nothing but a thinking thing," later emphasizing that he might have a body or extension, but that it is not essential to his being (Descartes 65). Hence, the intellect must be epistemically prior to both God and extension: obviously, Descartes would hardly agree that the intellect is ontologically prior, since the existence of everything is maintained by God. To argue he is a realist, however, all that is necessary is to establish that the existence of the intellect comes before the existence of extension as far as Descartes is concerned. Because mathematical objects are only known by the intellect and not by the body or its faculties-- as Descartes demonstrates in the intellect understanding the chillagon, a thousand-sided object, but the imagination (a faculty that has to do with extension and the body) being unable to distinguish it from any other many-sided figure-- the special status of certainty given to the intellect might be extended to its objects (Descartes 92).

In addition to this, Descartes also places emphasis of the truth of mathematical propositions repeatedly throughout the *Meditations*, as that which is indubitable: towards the beginning, recognizing mathematics to “contain something certain and indubitable,” and later giving them the status of that which is “clear and distinct,” on par with the concept of God in Descartes epistemology (Descartes 61, 70). So, mathematical propositions are both necessary truths and innate to humans, and this is so because mathematical objects have a “determinate nature, essence, or form, which is unchangeable and eternal… and which does not depend on my mind,” demonstrating that mathematical objects are not dependent on any intellect for anything true to be said of them (Descartes 88). There are two sources of immutable essence for any given thing: either this essence is derived from a mind, having defined and thought of its essence, or the thing exists and its essence comes from the intellect’s understanding of it. Descartes tells us that mathematical objects are not dependent on any human intellect, hence they must exist if they have an essence to which some things are contradictory.

This is what one might deduce from Descartes’s fragmented asides to existence, truth, and mathematical objects in the *Meditations*, but not something explicitly said. Rather, he seems determined to not give any definitive statement on the existence of mathematical objects. The preponderance of the evidence, along with his insistence not to be seen as a realist, is on the side of concluding that he is, in fact, not a mathematical realist. While the source of the essence of mathematical objects is not a question Descartes answered, from his discussion of existence and essence in the Fifth Meditation it is hard to believe he would want to conclude that their essence is sourced from real mathematical objects somewhere. This question was simply not his focus.

In the debate that the Enlightenment Era is usually set up as, between rationalists and empiricists, Immanuel Kant is usually presented as the synthesis position, muddying already unclear lines of who can be considered a mathematical realist in a post-Platonic world. Kant has more to say on mathematics than Descartes, which, ideally (no pun intended), should make parsing his position on mathematical realism an easier task. One would be remiss to talk about Kant, though, without recalling David Hume, the Scottish philosopher who Kant credits with "waking him from his dogmatic slumber," and a staunch empiricist of the modern era (Hatfield xiv). Hume's radical skepticism of the so-called rational principles taken for granted by many philosophers before him, such as Descartes, is what makes his philosophy revolutionary to modern epistemology.

Before discussing the perceived failures of his predecessors to provide a sufficient philosophy of mathematics, however, it would be prudent to discuss the elements of Kant's epistemology more generally. The term "intuition" pulls a lot of weight in Kant's epistemology, and cannot be taken in the contemporary common-sense meaning of the word, but describes "a mental representation that is particular… and presents objects concretely" (Hatfield xxiv). Hence, Kant, and Kantian scholars, repeatedly talk about representing mathematical concepts, which are the abstract notions, concretely in the intuition. For example, the concept of an isoceles triangle might be represented concretely in the intuition by imagining a particular isoceles triangle in the intuition of space. The pure intuitions that Kant invokes most frequently, the material from which these concrete images of mathematical concepts are fashioned, are space and time. His focus is more so on the intuition of space, since it underpins the geometrical concepts that are most familiar in mathematics, and most revered, thanks to the fame of Euclid.

The other important aspect of Kantian epistemology underpinning his philosophy of mathematics, which informs how he builds upon his predecessors, is the distinction between analytic and synthetic judgements, a distinction that cuts roughly along the same lines as the methods of analysis and synthesis discussed prior. An analytic judgement would be one that comes from the bare definitions of the terms involved: famously, "a bachelor is an unmarried man." A synthetic judgement is one that requires building up more than the definitions of the concepts themselves and “augments [one’s] cognition, since it adds something to [the] concept” (Kant 16). For example, constructing a mathematical object *in concreto* in the intuition would lead one to a synthetic judgement. These types of judgement are further modified by Kant to be either *a priori* or *a posteriori*, an uncontroversial distinction dating as far back as Plato's divided line, and his distinction between the realm of the Forms and the sensible world, respectively. It falls out with relatively little argument that there cannot be judgements that are both analytic and *a posteriori*, as that which strictly defines a thing cannot be exhibited in the phenomenal world. Even if the subject were empirical, one “need[s] no further experience outside [their] concept of gold” to analytically say that gold is a yellow metal (Kant 17). Hence, no *a posteriori* judgements can be analytic, leaving three other types of judgements that can be made under Kantian epistemology.

Schematically, the types of cognition, and the discipline that Kant considers to be typical of each type, can be represented as follows:

	A priori	**A posteriori**
Analytic	Metaphysics	Ø

Synthetic	Mathematics	Natural Sciences

This answers the question of why Kant believed the views of philosophers before him, such as Descartes and Leibniz, which he terms the "mathematicians' view" and the "metaphysicians' view" respectively, to be inadequate (Shabel 40). The mathematicians' view, that of Descartes and Newton, tries to meet the apriority and applicability demands by claiming that mathematical cognition is "a clear perception of the mathematical features of the extramental natural world" that is "illuminated by… reason" (Shabel 40). On first look, recognizing the mathematical features of the empirical world would satisfy the applicability demand, and the natural light of reason would satisfy the apriority demand. However, that would necessitate accepting the natural light of reason to be a sufficient argument, which many do not, including Kant, causing the mathematicians' position to fail the apriority demand. Hence, if one does not accept the conditions that let the mathematicians' view meet the apriority demand, then the position sounds familiar to that of a modern scientist: the universality of pure mathematics is a lower concern than its applicability to their specific field.

The metaphysicians' position, which Shabel identifies with Leibniz and Wolff[7], identifies our "pure mathematical knowledge [with] formal knowledge of the logic of mathematical relations," allowing for a purely formal understanding of mathematics (Shabel 42). The practice *dojour* during the modern era was Descartes's method of "constructible equations," where there was constant reference to the geometrical figures that algebraic equations were modeling, since the problem only has a solution once it is constructed geometrically. On a formalist stance, however, it is superfluous to know the referents of the symbols being manipulated. In fact, the

[7] Christian Wolff, a mathematician and philosopher whose textbooks Kant used when teaching, the examination of which lends interesting insight into Kant's view of contemporary mathematics, especially Descartes's method of "constructible equations."

symbols and the rules for their manipulation themselves was part of pure mathematical knowledge on the metaphysicians' view. Today, this branch of mathematics is known as algebra-- but it is incongruous with mathematical practice at the time, where algebra was a method by which geometric or arithmetic problems could be solved, but not a discipline in itself. To prevent our argument from temporal chauvinism, then, we must say the formalist view does not properly describe mathematics at the time in which they argued. To take the symbol of Descartes's constructed equations to be mathematical objects, and to take the manipulations of these equations to be mathematical cognition in itself, is unfaithful both to the historical conditions of their development, as well as the way they are used by mathematicians generally at the time. Hence, the metaphysicians cannot fully describe mathematics, and since they take these symbols to be meaningless, they cannot apply meanings to apply them to other realms, geometrical or empirical. In the absence of meanings, the symbols cannot be applied, and the formalist stance fails on the applicability demand.

The mathematicians' and metaphysicians' positions, as Kant conceives of them, can be seen as corresponding with a view of mathematics as *a posteriori* synthetic cognition or *a priori* analytic cognition, respectively. To meet both the apriority and applicability demand, then, Kant is motivated to claim that mathematics is *a priori* synthetic cognition. The necessary, and most controversial, piece of his philosophy of mathematics then becomes the constructability of mathematical objects in the pure forms of intuition, which more generally is his doctrine of Transcendental Idealism (Shabel 44). This does not answer the question of Kant's ontological and metaphysical commitments as far as mathematical objects are concerned, however.

In section 32 of the *Prolegomena* is where the distinction between *phaenomena*, "sensible beings or appearances," and *noumena*, "special intelligible beings," becomes important in

Kantian philosophy, where by intelligible he means that the *noumena* cannot be accessed through the senses (Kant 66). The names by which he refers to sensible and intelligible beings is fitting for how they function in his philosophy: *phaenomena* are "mere" appearance, and only tell us about how our senses are affected, whereas "we do not and cannot know anything determinate about" *noumena* (Kant 67). Thus, objects of cognition, including mathematical objects, fall under *phaenomena*, which are the objects of possible experience, leaving Kant agnostic on the question of whether there is a *real* referent of the mathematical objects constructed in the pure intuition, since he is agnostic on "things in themselves," that are "independent of… both our senses and understanding," making it impossible to know anything about them[8], including whether they exist or not (Kant 74).

Since Kant is agnostic on *noumena*, if he is to have an ontology at all, it is not of things in themselves, but an ontology on the basis of our possible experience[9]. This does *not* preclude him from "capital T truths" or "the search for diamonds," as Trudeau might put it in the terms of his Diamond Theory of Truth (Trudeau 114). The "truths" that Kant is seeking are still objective and necessary, which classifies them as diamonds, though not *about* "things in themselves,". When we restrict our analysis to his synthetic *a priori* statements, those that are the meat-and-potatoes of mathematics for Kant, then the question of *noumena* and truths of it becomes superfluous: "synthetic a priori statements were his diamonds" (Trudeau 114). To say mathematical propositions are truths does not imply Kant is a mathematical realist, but to say mathematical propositions are *truths about the world*, even if that world is that of our possible experience,

[8] There is an interesting paradox here regarding how it is possible to know that *noumena* are unknowable at all. Self-reference paradoxes such as these will be dealt with briefly in the discussion of Gödel's incompleteness proof.

[9] This "weaker" sense of ontology, originating from Kant's admittance than we cannot conceive of things unconditioned by the forms of sensibility, is a new development from his position that signifies a major shift in metaphysical debates, and is mirrored (as will be seen) in Quine's position. All this is to say, Kant is an extremely influential philosopher, even in the tradition of mathematical realism.

would be sufficient to call him a mathematical realist, for what Kant takes to be valid ontological claims (restricted to *phaenomena*). In Trudeau's view, Kant is indeed making the claim that mathematical propositions are truths about the world, based on how Kant conceives of the form of sensibility that is space: "the space studied in geometry… is pervaded by Euclid's Postulates because they are the very principles by which they are organized" (Trudeau 113). In other words, it is impossible to experience space any way *but* Euclidean, and more generally, it is impossible to experience anything at all except mathematically.

To further elucidate Kant's view as charitably as possible, we draw a distinction between the rules of mathematics, the laws by which we experience the world, and the articulated mathematics, for example, Euclid's Five Postulates. The challenge, then, is to construct a coherent, Kantian story to take us from the former to the latter. The rules structure the forms of sensibility, including the pure form of sensibility of space, and are the "formal conditions of *a priori* intuition" (Kant 180). Since the pure intuition conditions our possible experience, our minds are drawn to recognize these mathematical objects and propositions in our actual experience: while "we require no experience thereto at all," without this actual sense data that supports mathematical cognition, mathematics "would amount to nothing but… a mere brain phantom" (Kant 180). In experiencing this supporting, sensible data, we construct a more abstract and mathematically formalized object *in concreto* in the pure form of sensibility, and this allows us to create the actually articulated rules. To explain by example, consider triangles. Since our experience is conditioned by the form of space, which is governed by innate, mathematical rules, we are drawn towards seeing triangles out in nature: the shape formed by laying a ladder against a wall, the shape formed by a diagonal across a square courtyard, the inner petals of a flower. In seeing this supporting data of our intuitions, over and over, we construct the abstracted triangles, divorced from the *a posteriori* in the form of space, and recognize certain properties of them. In so doing, and in attempting to capture these abstracted, constructed entities, we articulate a definition of triangle: a shape with three sides. The source of these objects, then, is still an innate and necessary rule, "pre-baked" into our minds. The "formal conditions" of our possible experience could not be different, without our experience being radically different, than what they are for Kant, and could not lead to a substantively different mathematics. Of course, our language might be different than it is, due to historical factors, but the actual underlying mathematics would be the same. To harken back to the alien example from chapter 1, the aliens' mathematics would be translatable to our mathematics for Kant. The only divergence would be at the point of articulation into linguistic rules, such as the actual words for definitions or axioms.

On this interpretation, Kant remains very much in the same vein as mathematical realists we have already seen, subject to the restriction to the realm of possible experience. The rules of synthesis, captured by mathematical propositions are causally efficacious, but only insofar as they structure the conditions necessary for our possible experience, and they are necessary truths, insofar as it is inconceivable within the bounds of our possible experience for us to conceive of mathematics differently. Mathematical entities are also separable from the *a posteriori* entirely, but not from materiality necessarily: if we are to interpret the oft-repeated phrase of Kantians, "constructing in the intuition *in concreto*," the most natural interpretation for geometric objects might be that they are "constructed" in some sense out of space. The precise process by which mathematical entities are constructed for Kant might be left to other scholars,[10] but for our purposes, places him along lines more similar to al-Biruni in terms of separability. Even along

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[10] Shabel (1998) devotes some time to untangling what Kant means by such constructions, and how to construct (pun unintended) a consistent position of Kant's mathematical constructions that can be extended to arithmetic, since he spends the majority of his time talking about mathematics referring to geometry. This is unsurprising given the historical context for mathematics at the time, which Shabel also delves into.

the criteria proposed at the beginning of this chapter, Kant provides a theory of mathematical propositions as being necessary and innate. Hence, while Kant might be seen as the synthesis position in the empiricist and rationalist debate, I interpret him staunchly as a mathematical realist, from several angles presented here.

Again, according to Shabel's analysis, Kant meets both the apriority and applicability demand. Kant's theory of mathematics places mathematical entities as central to the possibility of our experience, as our spatial and temporal senses are inherently mathematical, explaining why mathematics is so unreasonably effective at predicting the empirical world. He meets the apriority demand as these mathematical principles are hardwired into our minds, to put it crudely, there is no way mathematics could have ended up differently: it's simply outside the realm of our possible experience, then, for space, and therefore geometry, to not be Euclidean. This is why Trudeau puts the philosophical implications of non-Euclidean geometry in dialectic with Kant in *The Non-Euclidean Revolution*. The possibility of a logically consistent geometry that contradicts Euclidean geometry would be repugnant to Kant's sensibilities, and Trudeau argues this counts against Kant's mathematical realism, since hindsight is 20/20.

Since Euclid's *Elements* became widespread and hailed as the gold standard of logical reasoning, geometry was seen as the bedrock of mathematics, much in the way we currently see algebra or arithmetic to be the most important thing one can get out of their mathematical education. Alongside the rigorous reasoning of the *Elements*, however, was an assumption that this was simply how the world worked: an idea that Kant enshrined in his philosophy, and a concept that Trudeau calls a material axiomatic system. If we see Euclidean Geometry as a material axiomatic system, then the axioms should be easily accepted as trivial facts of how the world works. In other words, they should psychologically be an easy pill to swallow. This is not the case, for many mathematicians and philosophers throughout the ages, with Euclid's Fifth Postulate. The fifth postulate is the parallel postulate, and while not originally formulated in this way, is equivalent to the oft-repeated phrase "given a line and a point, there is only one parallel line going through this point," also equivalent to there being 180° in a triangle. To many, this claim seems like it ought to require a proof, and so they set out to acquire one. The usual way of doing this was with a proof by contradiction: assume something contrary to the fifth postulate, and attempt to derive a contradiction using the other four postulates. In doing so, mathematicians found something unusual.

Most notably, Carl Friedrich Gauss, arguably one of the greatest mathematicians in history, started to develop non-Euclidean geometry in 1813, less than ten years after Kant's death, in spite of not publishing anything on it during his lifetime. He was amongst the first mathematicians with the idea that it may not be possible to derive a contradiction: rather, they've created a contradictory, and independently consistent, new geometry, called hyperbolic geometry[11]. This was further developed by mathematicians Lobachevsky and Bolyai independently, between 1829 and 1832, into a consistent system. This could no longer be referred to as a material axiomatic system. Rather than describing what the world *is*, the way Euclid did, non-Euclidean geometry is simply following the logical trail where it leads, to a different system that Trudeau calls a formal axiomatic system. It seems that while Kant met both the apriority and applicability demands with his philosophy, the mathematical developments almost immediately proceeding his death favored a philosophy that emphasized a formalist

[11] The terminology is attributed to Felix Klein, another pre-eminent mathematician, along with the term "elliptic geometry" for the other kind of non-Euclidean geometry. The multiple contradictory geometries comes from negating the fifth postulate: either there can be more than one parallel line through the point, or there might be no parallel lines through the point, and each generates a different geometry. Hyperbolic geometry is the latter of these two, as there is an abundance of such lines.

stance and the apriority demand over the applicability demand. This represents another great shift in the thread of mathematical realism: what Kant did to push the realist tradition forward, the non-Euclidean fiasco (arguably) undid.

It is in this context that Gottlob Frege advanced his logicist thesis. Following the downfall of geometry as an indubitable science, arithmetic was seen as the more stable foundation: attributed to Gauss, "Mathematics is the queen of the sciences-- and number theory is the queen of mathematics." Mathematicians and philosophers were burnt once, however, and arithmetic did not seem to be *ipso facto* a more stable option that truly caught the essence of mathematics. Instead, mathematicians delved down deeper to what *is* indubitable and, if not essential to the world or our experience of it, essential to mathematics: logic. Hence, Frege, hailed as one of the most influential logicians since Aristotle, turned to logic in his pursuit to save mathematical realism.

Frege's most influential work is his *Foundations of Arithmetic*, in which he lays out his thoughts, pun unintended, for a new logical calculus. In it, he contrasts himself explicitly with Kant, and places himself with the position that Shabel ascribes to the metaphysicians, Leibniz and Wolff. Namely, that mathematical propositions are analytic *a priori* rather than synthetic *a priori*. He justifies this move by claiming that Kant's view of what is analytic is much too restricted: for Frege, analytic judgements are those that are "deducible solely from purely logical laws" with no extra-logical premises used in such deductions (Frege §90). This is precisely what his logicist project is, to reduce mathematics to logical moves and place it on a more stable foundation than it has been shown to have this far. The fact that mathematics can shift and change after a "period of mathematical digestion raises unsettling doubts about mathematical certainty," and grounding it on that which Frege saw as unchanging was a solution (Wilson, "Royal Road" 150).

Frege, taking a very endoxic approach, recognizes a problem in philosophy of mathematics that has been talked about many, many times over: how it is we can count things, more specifically, by counting of units. As Frege quotes Descartes saying, "the… plurality in things arises from their diversity," but "units are units in respect of being perfectly similar to each other," as he quotes from Jevons (Frege §35). There is a problem: can we count things as being different while being perfectly the same? If, instead, units were different, counting would be ambiguous and fall into further problems. For instance, if Lawrence were to open the freezer to check how many chicken thighs there were left for dinner, he is engaged in this very problem of counting: either the things he is counting are entirely identical, or they are not. If it were the case that they were perfectly identical, his "chicken thigh unit" can only be a single thing, for only one chicken thigh can occupy that space at that point of time. In which case, he cannot count multiple chicken thighs, as they are all different, say (chicken thigh), (chicken thigh)', (chicken thigh)'', and so on. In other words, "the result runs perpetually together into one, and we never reach a plurality" (Frege §39). On the other hand, if Lawrence's counting units are not required to be identical to each other, then he might count the plastic bag the frozen chicken thighs came packaged in as well as the chicken thighs themselves, and say there are two: in which case, his wife will go to cook dinner, see a single chicken thigh, and get very annoyed with him. Put in different words, "the result is an agglomeration… of objects… still in possession of precisely those properties which served to distinguish them" (Frege §39). There is some intermediate required, then, for Lawrence to accurately count the number of frozen chicken thighs in his freezer.

Frege solves this problem with his key distinction between concept and object, and “only concept words can form a plural” (Frege §38). The distinction he draws follows Aristotle’s distinction between categories and substances closely: a concept is “the reference of a predicate,” and Aristotle similarly calls the categories that which is predicated to substances, while an object “can never be the whole reference of a predicate, but can be the reference of a predicate” (Frege, “Concept” 173). So, to solve Lawrence’s chicken thigh problem, we must consider the concept of “chicken thigh in Lawrence’s freezer,” and notice how many objects “fall under” this concept. We can then say that the number three falls under the concept “chicken thigh in Lawrence’s freezer,” and this solves similar problems raised and similarly solved by Aristotle with counting an army: there is a single *army*, but thousands of *men.* Recast in Frege’s terminology, the object one falls under the concept of army, and under the concept of man falls ten thousand. Hence, after having several concepts under which four falls, we abstract four on its own to give it to us as a Number, so “abstraction does… precede the formation of a judgement of number” (Frege §48).

This does not give Frege a robust enough theory of concepts and objects to fully describe numbers to logically deduce mathematical propositions. Notice, for instance, that the concept “chicken thighs in Lawrence’s freezer” can be further decomposed into simpler concepts, say, “chicken thighs” and “that which is in Lawrence’s freezer,” which are marks of the concept “chicken thighs in Lawrence’s freezer” (Frege, “Concept” 177). These marks of concepts are themselves concepts, under which the original concept is subordinate: “chicken thighs in Lawrence’s freezer” falls under the *second-order concept* of “chicken thighs” and the second-order concept “that which is in Lawrence’s freezer.”

Imagine, though, there was a zombie pandemic that spread over the entire world, and killed the power everywhere except in Lawrence's house. Then, those objects falling under "frozen chicken thighs" and "frozen chicken thighs in Lawrence's freezer" would be the same objects-- the two concepts would be mutually subordinate, a relationship between concepts that is "closely linked" to identity between objects (Schirn 988). In the absence of his technical work, since his *Foundations* is a meta-logical work, to account for his use of the definite article "the" when saying "the concept *F*," he must introduce extensions of concepts, which serve as the set of objects *x* that make the predicative statement *Fx* true. In less technical language, it is the set of objects that fall under *F*. This allows him, in *Foundations*, to talk about the *equinumerosity* of two extensions of concepts and formally define the bijection between sets, and thereby define a Number to be the equivalence class of all such equinumerous extensions of sets through this mutual subordination under equinumerosity (Frege §72). Notice, equivalence relations (and therefore equivalence classes) are purely logical for Frege as well, and existing in all logical systems, so his construction of numbers remains pure.

To determine whether Frege is a mathematical realist, however, there are two issues that remain to be cleared: what kinds of things are Frege's numbers, and what ontological commitments does he have to these things? In "Concept and Object," Frege makes it clear that the distinction between the two logical entities is paramount to him, so which is it that number is? In the very same paper, he refers to 2 as an object falling under the concept "positive whole number less than 10," and proceeds to show an argument that takes there to be two ways in which we might understand the number 4 (as either a concept or a concept-object) to be untenable and undemonstrative of his position (Frege, "Concept" 177). Rather, he takes the distinction he is arguing against to really be talking about the difference between the sense and the reference of number 4: the former being the intelligibility and semantic content of the number 4, and the latter being the actual existence of the number 4. For instance, Mark Twain and Samuel Clemens have separate senses, as one may know of Mark Twain as the writer of *Huckleberry Finn*, but not know it's a pen name, but Mark Twain and Samuel Clemens have the same reference, as they are in actuality the same man. So, it seems numbers must be logical objects for Frege, and "asserts something objective of a concept," hinting that it might have to do with the reference rather than sense of an object, the latter being subjective, which leads one to believe he does have some positive ontological commitments to numbers (Frege §106). Indeed, the way that Frege refers to numerical propositions, such as the Pythagorean Theorem, as "timelessly true, true independently of whether anyone takes it to be true," much in the way that a newly discovered planet "already before anyone has seen it, has been in interaction with other planets" (Frege, "Thought" 302). The analogy, however casual it might be, between discovery of scientific truths and the discovery of mathematical truths, heavily indicates that Frege had realist leanings. In addition, he places Truth as a burden of the logical sciences, and Truth and Falsity as "the only objects that belong intrinsically to the fundamental parts of logic" (Schirn 1001).

Since mathematical propositions are necessary as logical constructions, one step removed from abstractions of concepts in the real world, Frege meets the applicability demand. He also meets the apriority demand easily, since after this abstraction from concepts, the construction of mathematical entities is logically pure and analytic. Hence, he meets both demands and the criteria for a mathematical realist, and it is obvious from his writings on the onus of logic to determine and define Truth, along with his distinctions between subjective and objective judgements (of which mathematics is the latter) that there is a strong case to consider Frege a mathematical realist. In fact, one might claim that "[realism] reappear[s] in twentieth-century surveys of the philosophy of mathematics under the new [name] logicism," as the logicist thesis commits philosophers to a slew of mathematical entities, positing the existence of mathematical entity upon entity (Quine 14). If we are to accept this claim, then the logicist project in general is a realist project as well, including the continuation of Frege's project by Bertrand Russell, who found a paradox in Frege's technical work, and Alfred North Whitehead in their *Principia Mathematica*, that aims to reduce all of mathematics to logic.

These logicist ideals, if not the entire logicist thesis, became part of the mathematical community's philosophical conversation, especially understanding of the foundations of mathematics through set theory and logic. Indeed, some of Hilbert's famous 23 problems, posed by the eminent mathematician in 1900, have to do with proving the completeness and consistency of mathematics. Hilbert, largely considered to be the founder of the formalist philosophy of mathematics[12], believed that mathematics could be shown to be complete and consistent, using the logical tools that the logistics had started to develop with Frege, or shown to be incomplete or inconsistent. This optimism is a result of the bivalence (otherwise known as the principle of the excluded middle) that was assumed by logicians and mathematicians at the beginning of the twentieth century: for any given statement in an axiomatic system, either the statement can be proved or its negation can be proved, making it both complete, and if the or is taken to be exclusive, consistent. Hence, the truth value of any proposition can be determined, and no contradictions are derivable within the axiomatic system.

The solution to Hilbert's completeness and consistency problems were answered by Kurt Gödel, an Austro-Hungarian mathematician and logician, with his well-known Incompleteness

[12] The formalist philosophy, summarized, holds that mathematics is really just the rules by which we manipulate uninterpreted symbols on a page: as such, truth or falsity is strictly a matter of consistency to a formalist such as Hilbert.

Theorems that arguably undermined Hilbert's project. His first incompleteness theorems states that there are arithmetic statements that are "neither provable nor refutable… though true in the standard model," which contradicts the bivalence of provability previously stated, and his second incompleteness theorem states that the consistency of an axiomatic system cannot be shown within the system itself (Kennedy). While the technical elements of his first proof are interesting and built a basis for much of mathematical logic today, through his construction of self-referential Gödel-statements, this is not the most interesting effect of Gödel's incompleteness theorems for our purposes: rather, the realist implications of interpreting his theorems is the issue we must focus on, more specifically the philosophical nature of truth in mathematics.

The controversy of the first incompleteness theorem is derived from the latter part of the statement, namely that the proposition in question is "true in the standard model," a caveat that is, in the strictest sense, superfluous to answering Hilbert's question of the completeness of axiomatic systems: simply stating that there are statements whose truth values in the system are undecidable would be sufficient. I believe that this clause was to demonstrate Gödel's own platonist view, in which the truth of a mathematical proposition is decidable on a basis besides the strict logical calculus, in contrast to Hilbert's formalist view, where consistency is enough to impute "existence" (in some less strict sense of the word) of an axiom or the entities it posits. The result of a realist philosophy is that the truth of a statement must be able to be determined: "snow is white" is a true statement, for instance, if the snow truly is white. In mathematics, the truth of a statement for Gödel, then, must be something more fundamental than whether or not we can prove it within an axiomatic system, as he shows this to not always be possible.

Since he is a platonist, by Kennedy's standards, we might assume Gödel thinks the truth of a mathematical statement really comes from the Forms (or intermediates) of these mathematical entities interacting with each other. So, if our axiomatic systems are unable to prove the truths that can be grasped by appeal to non-sensible entities, then the realist's upshot from Gödel's incompleteness theorem might very well be that axiomatic systems simply do not capture the true mathematics. Hence, while Gödel's platonist philosophy and incompleteness theorems might be seen as contrary to one another from a Hilbertian perspective, there is still a sense in which he can be taken to not undermine the realist thesis.

W. V. O. Quine, an American philosopher, logician, and mathematician who worked in many parts of analytic philosophy during the latter half of the twentieth century, is the seal upon the philosophers we are surveying, largely due to the lasting influence he has had on philosophy, so that most other positions are placing themselves in relation to Quine. In fact, I do not think I would be remiss to say that all of twenty-first century analytic philosophy is simply a footnote to Quine. Part of the revolutionary nature of his work, his rejection of the analytic-synthetic distinction, is directly in conversation with Hume's distinction between relations of ideas and matters of facts, and Kant's analytic distinction which, shown by Frege's acceptance of the dichotomy, was all but ubiquitous. He denies this distinction by examining the typical analytic statement: "every unmarried man is a bachelor," which, for a proponent of the distinction, follows immediately from the definition of bachelor or unmarried man. However, as a philosopher of language, Quine does not take this lightly. The natural question to him is "how do we find that 'bachelor' is defined as 'unmarried man'?" noting that an observed similarity is the basis for a lexicographer calling them synonyms, making it inappropriate as an analytic statement (Quine *Two Dogmas* 24). He reduces this problem of definition to one of "cognitive synonymy" that allows two concepts to be interchanged without changing the meaning of the sentence itself, though this leads one to another question of meaning, and how to assign this in a way that does not appeal to any empirical data (Quine *Two Dogmas* 28). This ends up being impractical, especially in an extensional language: excusing the technical aspects, he concludes that the distinction between analytic and synthetic statements is untenable.

His philosophy of mathematics, then, must fall outside of the taxonomy given to us by Kant that neatly categorizes thinkers from Descartes to Frege. He gives an explicit criterion by which one can ascribe ontological commitments to a theory generally: "by our use of bound variables," where he is referring to existential quantifiers like *for all* and *there exists* (Quine 12). More specifically, we commit ourselves to the positive existence of certain entities when they render one of our statements containing bound variables true. This leads him to conclude that classical mathematics "is up to its neck in commitments to an ontology of abstract entities," as it is a theory rendering statements such as 'there are primes larger than a million' true, witnessed by a number such as 1,000,003 and many such others (Quine 13). He concedes that this discussion does not answer the questions of his own ontology, as "what there is is another question" (Quine 16).

Finally, we are in a position to discern his criterion for an ontology we ought to accept rather than a linguistic criterion. His entry point is observing that we adopt ontology in a way not dissimilar to how we accept a scientific theory: keeping in the tradition of empirical skepticism, we can never say something certain and indefeasible about the out-there that is filtered through our less than perfect senses. This affects both the scientific theories we can develop as well as our ontologies, so the best we can do is adopt "the simplest conceptual scheme into which the disordered fragments of our raw experience can be fitted and arranged" (Quine 16). He expands upon the considerations involved in adopting a scientific theory, such as the existence of molecules, as being divided into five: simplicity, familiarity of principle, scope, fecundity, and predictive power (Quine *Posits* 76-77). He calls these all pragmatic considerations, and this forms his criteria by which we ought to adopt an ontology or a science. While we all have a "scientific heritage" and "barrage of sensory stimulation," the considerations by which we modify this inherited web of beliefs is "where rational, pragmatic" (Quine *Two Dogmas* 46).

However, one may argue that the utility of a theory, as encapsulated by Quine's five considerations, is not enough to allow for the existence of the entities it posits. Rather, we could simply use the theories for their pragmatic uses but not commit ourselves to any ontology. What evidence, then, Quine would ask, do we have to believe in anything? Why do I believe that a copy of Quine's *From a Logical Point of View* is sitting to my left on a white table? Simply because it is the easiest way of explaining the brown, biconcave shape in my periphery and perception of whiteness underlying everything I'm looking at. In other words, all of our ontologies are just theories that best encapsulate sense data: to do away with one on the basis that it is pragmatic commits us to getting rid of all of them, leading to solipsism. The fact that our theories help us organize experience is exactly "what evidence is" (Quine *Posits* 80). Just as we must accept the existence of molecules to explain our experience, we must accept the abstract entities of mathematics that are necessary for our scientific theories. This is the basis of the Indispensability thesis, an argument for mathematical realism based on its indispensability to our current science, which Hilary Putnam was also a proponent of.

To make Quine's position more radical, he does not fit within the apriority and applicability demands in the way the rest of our thinkers have. He makes no claims about whether or not the truths of mathematics are necessary or not, and his argument does not hinge on whether Euclidean or non-Euclidean geometry is real. Rather, what is necessary is that we accept the mathematics that is necessary for our science. It might be extrapolated, then, that either Euclidean and non-Euclidean geometry could be true in Quine's view, depending on what scientific theory we consider indispensable to organizing our experience. If classical mechanics is that which best models our experience, as was thought during Galileo's time, for instance, we must accept Euclidean geometry to be true. Conversely, if we consider relativistic physics to best satisfy Quine's five considerations, a non-Euclidean geometry would be true and indispensable to our understanding of space. From this, it is clear that Quine's philosophy meets the applicability theory, and seems almost tailored towards it with his pragmatic criteria.

While he certainly posits the existence of matematical entities, is this sufficient to call him a *mathematical* realist? I do not think so. His ambivalence on the necessity of mathematical theories to be the way they are is an indicator of this: in fact, his theory does not necessarily lead him to conclude that mathematics is real. Even if we are to accept his argument as it is, which we shall for the purposes of understanding what kind of realist he is, Quine says nothing that is essential to mathematical theories or entities that convinces us to adopt them into our ontologies. Rather, he only makes arguments that they are pragmatic to accept into our ontologies. This is his criteria for adoption, so this is unsurprising. However, we must then conclude that the only reason we accept mathematical entities into our Quinian ontology is their indispensibility. If Quine were (still alive, and) transported back in time to pre-Galilean times, his "current science" would not be dependent on mathematics, and it would not be adopted into his ontology at all. To be clear: this does not imply that Quine is inconsistent, nor is this an argument against Quine's position. Instead, it is an argument to not classify him as a mathematical realist *per se*, but rather as a pragmatic realist and accidentally as a mathematical realist.

Chapter 3: The Twelve-Step Realist Program

The goal of this chapter is, given the survey of mathematical realism through the ages that we have conducted, to construct the most defensible position for a mathematical realist to take. We will look to the thinkers we classified as realists prior, and formulate what is essential to a realist position. Then, we will, standing on the shoulders of these giants, articulate the strongest argument for each piece of this mathematical realism and form a cohesive position from it. Finally, I will give reasons why one ought to take this position if they think mathematical realism is an attractive position, specifically if they are an analytic philosopher with similar inclinations to the thinkers explored thus far, or take it seriously if they are unsympathetic to mathematical realism. My goal is that one should finish this chapter willing to accept this version of mathematical realism and agree it is the most defensible version we can construct.

First, we note that the realist positions explored fall into two major camps that partition the set: materialists and poly-ists[13]. By a materialist, I mean a position in which the primary elements of ontology are material things, and the realm that holds answers for us is the perceptual or empirical world. Further, in this context, evidence for the existence of mathematical objects would be empirical and accessible in the perceptual realm. For the materialist, while they may posit other realms for their ontology, their argument for a positive

[13] I hasten to add that I do not intend to construct and later take down a false dichotomy: the idealist position has indeed been omitted. This is, for one part, due to the fact that we have not surveyed anyone I would (or popular consensus in the philosophical community) consider to be an idealist, and the purpose of this chapter is to draw on what we have previously presented in Chapters 1 and 2. For another part, I omit idealism because it cannot meaningfully address the applicability demand, as idealists deny the perceptual realm entirely, in which applicability can even be discussed. In addition, I hold that idealism has fundamental problems with its epistemology. As I will argue later in this chapter, denying what is epistemologically prior and fundamental to us, our sensory data, makes for a bad entry point for any philosophical position, as it asks those who you hope would agree with you to dismiss their own intuition; specifically, that tangible, sensible things around me are real. For all these reasons, I do not give idealism any serious consideration here.

ontology of mathematical objects hinges upon the sensible one. Philosophers we have explored that I place in this camp would be the Pythagoreans and Ibn Sina. For both of these philosophies, the reality of mathematical objects is confirmed by their unreasonable effectiveness in explaining and arranging the barrage of perceptual data, which was previously addressed in Chapter 1 as the causal efficacy of mathematical objects, and in Chapter 2 as the applicability demand. Interesting to note, there are no mathematical realists from the modern or contemporary era that I put in this category, however, if one were to disagree and accept Quine as a mathematical realist, I would also consider his to be a materialist philosophy. In addition, I would consider Aristotle and Hume to be materialists as well, though not mathematical realists. I categorize these three philosophers as such because their substantive ontological claims are based on and stay in the perceptual realm. A distinction between materialists and poly-ists is not necessarily their entry point, as both may start in the perceptual, but what is foundational to their claims is the basis of their difference.

For instance, Quine could say sets are real, but he can say this if and only if the sensory realm continues to support a scientific theory that requires set theory (somewhere along the chain of indispensability): therefore, his acceptance of sets is contingent upon the perceptual realm. For Aristotle, he allows the forms to exist (though, not substantively), but only insofar as they are inhered in material things: in fact, forms cannot exist at all except by being drawn (or abstracted) from the perceptual realm. Finally, regarding Hume, while he has relations of ideas, which are true regardless of whether or not the subjects of such propositions exist or not, they do not form the basis of any substantive ontological claims: mathematics does not have applicability on his view, and any empirical is a matter of fact, not a relation of ideas. As I will discuss later, I do not think Hume can even rightly hold this distinction, collapsing his entire position into an inductive, purely materialist one. In short: to be a materialist, we require that substantive ontological claims are based in the perceptual realm, and to be a materialist mathematical realist, we require that these substantive ontological claims entail a positive commitment to mathematical entities.

In a poly-ist position, in contrast to a materialist position, there are multiple realms upon which the ontology hinges, leading to multiple ways in which the existence of something can be realized, different for each posited realm. Often the perceptual realm that is primary for the materialist is secondary (or tertiary) to the poly-ist, as the other posited realms usually have a higher ontological standing than the perceptual realm. Specifically for a mathematical realist, then, a poly-ist would be tasked with saying how it is mathematical entities exist, and in which realm, allowing them to make stronger claims about the necessity and apriority of mathematics than a materialist might be able to, as we will demonstrate later in this chapter. Mathematics would also end up in an ontologically privileged place relative to perceptual data for a poly-ist. Mathematical realists I would identify as poly-ists are Plato, al-Biruni, Kant, and Frege, all of whom draw a distinction between a "purer" realm such as the Forms or the mental world, and a less distinct realm in which certainty cannot be assured, the perceptual realm. For their theories to be fully consistent for some, it requires positing a third realm as well, such as Plato's intermediates or the ontological status of God for al-Biruni, and for others, it requires distinctions within the perceptual realm as the only one accessible to us, such as Kant's between *a priori* judgments and *a posteriori* judgments. As we have seen with the materialists, there are poly-ists that are not mathematical realists, such as Descartes, who draws a distinction between the mind and the body, which are representative of our existence in the rational realm and the perceptual realm respectively.

Our task, as earnest mathematical realists, is to decide which of these two mutually exclusive positions we think best caters to our goal of affirming the reality of mathematical entities. We may prefer materialism for its simplicity and for being grounded in common-sense, the same reason that Aristotle eschewed Plato's dualism and went for the more scientific materialism. It is certainly preferable if we take Ocaam's razor as our criterion, as a materialist philosophy posits no more than what we can see, touch, or taste. On the other hand, we might prefer poly-ism for its versatility and the ability to make stronger claims that are "necessarily necessary," so to speak. While there is much more we commit ourselves to ontologically, it allows both for greater breadth and strength of claims that can be made.

I believe that for a sufficiently strong and defensible mathematical realism, we must adopt a poly-ism of some sort. Through poly-ism, we can meet both the apriority and applicability demands required for any strong philosophy of math, whereas materialism can do better on applicability, but worse on apriority: in short, poly-ism does better on applicability than materialism does on apriority, giving us reasons to prefer poly-ism. This is because, while materialism can easily deal with the applicability demand by appealing to the unreasonable effectiveness of mathematics in, say, predicting how one should shoot in billiards, it has a much more difficult time dealing with the apriority demand. As was a common theme in the throughline of realism, and indeed philosophy as a whole, it is very wise to be skeptical of our senses. Hence, if our primary source with which to make positive ontological claims is the perceptual world, only accessed through our faulty senses, there are no without-a-doubt truths we can discover. Kant observed this in making his *phaenomena*/*noumena* distinction, and solved it by introducing the mental realm and *a priori* judgements which were necessary truths about the world, at least as we perceived it. This is still a stronger claim than the materialist can make. A

Kantian can say “a bachelor is an unmarried man,” provided they are not in front of Quine’s grave, with absolute certainty. A materialist, however, can only say “bachelors tend to be unmarried men, and will probably continue to be unmarried men,” much in the same way that Hume can doubt whether the sun will rise tomorrow[14]. Their degree of certainty might be very high-- but it can never reach the deductive certainty of a poly-ist position. This is not an argument against materialist positions in general. Only in the context of our goal, to make substantive, positive, ontological claims about abstract entities such as mathematical objects, does it become troublesome to base our philosophy upon the superiority of the perceptual world, in which any perceived mathematical objects are inexact and improper to continue any deductive line of mathematical reasoning. For example, if we would like to reason out the value of π, defined as the ratio of the circumference of a circle to its diameter, looking to the empirical world will not give us the same answer every time, even for the “same” circles: measurement errors aside (of which there are many to make measuring in the empirical realm less than precise), no circle in the perceptual realm will be truly circular, giving us a lack of precision. Even if we are to increase our sample size and take the ratio of the circumference to the diameter of thousands, millions, or billions of circles, and average these all, we may come very close: but we would never be perfectly certain that this will work for the billion-and-first circle. In fact, as we defined π and our number systems generally more rigorously, we found it to be an irrational number,

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[14] It is important to note that this not what Hume would say of his own position: in the *Enquiry*, he does make a distinction between relations of ideas and matters of fact, which is the precursor to Kant’s distinction between the analytic and synthetic. However, Hume is also a radical empiricist. While “a bachelor is an unmarried man” would be a relation of ideas for Hume, his very distinction falls apart with Quine’s argument for the untenability of the analytic/synthetic distinction. We reinterpret Hume in this way, but allow Kant to hold his distinction, because Quine’s question of meaning (and subsequent tying of meaning to empirical data) is especially applicable to Hume, who sees the empirical as supreme, whereas Kant does not hold the empirical to be supreme and has some other recourse to *a priori* judgements in order to hold this distinction. Hence, I claim it is more consistent for Hume’s radical empiricism, which is seen as more essential to his philosophy than the distinction between matters of fact and relations of ideas, to say that “bachelors are unmarried men” is subject to the same uncertainty that the rising of the sun the next day is subject to.

rendering our previous method contradictory. I will speak more about definitions further in this chapter, but it remains to be said that our best estimate of π in the ancient world was not found by taking such ratios of many, many circles and then averaging these out. Our best estimate of π came from Archimedes's method of exhaustion[15], which did not rely on any specific circle, or any number of specific circles, but rather on abstract definitions and deductive arguments. Therefore, I believe we ought to, as earnest mathematical realists, adopt a poly-ist position.

The next stage of our project is to discern what kind of poly-ist position to take, and we have the advantage of having completed a historical survey of positions we might take. As mentioned, our models in this case are Plato, al-Biruni, Kant, and Frege. Rather than pitting great thinkers, all of whom were writing in extremely different contexts, against each other, I propose we determine what elements of their poly-ist, realist positions have stood the test of time and are common amongst them. I have identified five such themes that are present within at least two of the philosophies being considered: 1) multiple levels of perception and/or existence (what I referred to as poly-ism), 2) certainty correlated to existence, 3) mind independence, 4) the perceptual as a reflection of the real, and 5) the unchanging nature of mathematical objects. We will consider them each separately to determine which is essential to realism, and if so, what is the most defensible argument for each.

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[15] Archimedes's method of exhaustion to find π involved constructing regular polygons with an increasing number of sides, with the intuitive (and later, formalized) understanding that as the number of sides in a polygon increases, the better it approximates a circle. His method procdeeded as follows: inscribe a regular polygon inside of a circle, and another of the same number of sides enclosing the circle. Then, find the ratio of the distance from the vertex to the center of each regular polygon, and the circumference of each regular polygon (both of which would be easy to find given our knowledge of triangles). The value of π is less than this ratio for the larger polygon, and greater than this ratio for the smaller polygon. Hence, we can reach greater degrees of accuracy to calculate π by increasing the number of sides. As our idea of limits evolved, we were able to generalize this and approximate π by evaluating the limit of this ratio as the number of sides of the regular polygons approaches infinity.

1. Poly-ism

With respect to poly-ism, we have already argued that it is the most defensible version of realism, as opposed to materialism. Hence, must posit at least one other level of existence in order to ultimately justify the reality of mathematical objects. As far as the strongest argument goes, the entry point is, as mentioned, empirical skepticism, a theme that has been justified multiple times through the philosophies of those philosophers mentioned, one such argument being Plato's argument about the three fingers, of which one might be both short and tall relative to the other fingers. The purpose is to demonstrate that our senses are not stable means of determining the truth values of certain propositions; a pencil in a cup of water appears to be bent when it is not, and we see a person's face in patterns containing no such thing. The well-documented failure of our senses to capture experience in a consistent manner has generated the entire field of cognitive psychology. No proposition can be said to have a stable truth value based on that which we sense moment to moment, for instance, the proposition "the square of the hypotenuse of a right triangle is equal to the sum of the squares of its legs," which changes depending on how accurate our measurements are and whether or not the material triangle is sufficiently planar. As soon as we add another layer of reality which organizes the perceptual realm, these problems are fixed, and truth values are attributed based on this more stable reality. Rather than deciding whether the proposition is true for each situation, or only ascribing it a high probability of being true, we are able to say the proposition is true in all cases after proceeding deductively from the definition of a triangle and Euclid's axioms, all of which also hold their truth value in the stable realm.

The next question in defending our positing of a second, stable level of reality is why it is that we require truth-values to have the specific property of being fixed and bivalent. While it is certainly an assumption that most of these philosophers are operating on, that does not mean we can discount it as an axiom we may not be partial to if we, for instance, adopt an intuitionistic logic, and dismiss mathematical realism. Quine makes an argument for this assumption, in fact: if truth-values were able to fluctuate and were undecidable in certain cases, we would not be able to claim the truth of anything, and be in no better a position than solipsism. In addition to Quine's argument, to say anything at all, we must be able to say, for certain, the truth value of something: classically, this something is the principle of non-contradiction that generates a bivalent logic. Hence, if we agree that solipsism is not a tenable position to end up in, we must accept that there are stable truth-valued propositions. Since these cannot be found in the perceptual realm, they must be determined from elsewhere. Therefore, poly-ism of some kind is necessary. We have argued for poly-ism, but not for how many such levels must be posited. For simplicity's sake, we might posit only two: the perceptual and the truth-stable realm, for this is all we have convinced ourselves of thus far.

2. *Certainty and Reality*

This argument rests upon another theme I have identified, namely the positive correlation between the epistemological certainty of a proposition and our ontological commitment towards the proposition. We justify it being essential to realism since there is no other criterion by which we say that something exists, except by an epistemological one. To justify that this epistemological criterion ought to be certainty, there is a, mostly implicit, assumption that reality ought to be consistent. I claim this to be the case because we hold an intuitive understanding of the principle of non-contradiction: the epistemological certainty in this principle is what motivates our certainty in other purely epistemological propositions. We are more certain that proposition *A* is true when we are convinced that ~*A* is impossible. Hence, we are more able to convince ourselves of ontological propositions in this exact way. Putting this assumption up for argumentation puts us in the position of arguing for or against the principle of non-contradiction, an argument that seems ludicrous no matter where you land on it. In the interest of rigorous philosophy, currently acting as mathematical realists, we ought to *attempt* to justify our belief in the principle of non-contradiction. If we are to reject that reality is consistent, then, by affirming proposition *A* we cannot necessarily rule out ~*A* from being true, giving us a contradiction in reality. In any bivalent logical system, anything you may like could be derived from a contradiction, including that unicorns exist, for instance. This gives us, not only a crowded ontology, but one that does not do what we intend for an ontology to do for us, namely differentiate between what is and is not real.

Of course, this argument for the principle of non-contradiction begs the question: it assumes that the logical system contains the principle of non-contradiction, and also uses a *reducio ad absurdum* to push forward the argument. This would not convince someone staunchly against the principle of non-contradiction of its value, and such an argument would be futile anyway. We cannot agree on what is a valid argument at all if we do not agree on the principle of non-contradiction. Rather, as good mathematical realists we shall presently assume a bivalent logic. Then, if we agree reality ought to be consistent, commiting ourselves to a bivalent logic, our epistemological criterion for ontology must be that which best models consistency. Hence, deductive truths within a bivalent logical system hold the most consistency and are the most epistemologically certain to us, and therefore become the most real.

3. *Mind-Independence*

Next, we must examine whether the mind-independence of mathematical propositions is essential to mathematical realism, which we hope it might be as it was part of our criteria previously. First, though, we must answer the question of what it we mean by saying something is real, and in what sense we want to say that mathematical objects are real. While we are in agreement that mathematical objects are not part of the physical realm, we must be careful about what we mean when we say they have a mental existence, to distinguish it from the existence that a unicorn might have as a mental entity. In this case, it is necessary that we make some distinction between the category of mental entities that unicorns fall under and the category of mental entities that the Pythagorean Theorem falls under. Hence any mathematical realism requires such a distinction, and so this argument is essential to our ability to affirm the existence of mathematical entities.

This is exactly the distinction that both al-Biruni and Frege are attempting to capture by what Frege calls Thought and Idea. Ideas require a specific bearer in order for them to exist at all, and are therefore subjective, for instance the idea "Lawrence is good at ping-pong" must have a specific bearer, such as Murdoc, for it to exist at all with the caveat that Murdoc thinks it. Mark, on the other hand, might have a contrary idea, "Lawrence is bad at ping-pong," which would be just as believed to be true by Mark as the opposite is to Murdoc. Hence, an idea are those mental propositions that require a bearer and are, by nature, subjective. On the other hand, thoughts are those that need no specific bearer for them to have existence and be true, and are therefore objective truths. For instance, on Frege's view, whether or not there exists a specific person thinking of the Pythagorean theorem, the theorem exists and is true. He goes on to claim that the Pythagorean theorem was true even before anyone had thought of it.

Our challenge, then, is to evaluate Frege's claim: is it the case that the Pythagorean theorem is true in the absence of humans? It goes without saying that we formalized it into the algebraic equation that highschoolers memorize in their Geometry classes, but this is just the symbols. The theorem applies to things beyond us, but this assumes that reality is structured in a way that is translatable to our mathematics: if this is true, then we can accept Frege's position with no problem. The argument for its truth, however, is based upon something locked away from us, at least by way of our senses. Reality unfiltered through our senses is not something we can access, as it involves that which is outside the realm of possible experience as Kant argues.

For these reasons, since we are making a human-centric claim, we must take a human-centric position, and hence the Kantian position. The Pythagorean theorem, then, is a truth about reality as we experience it, so it is true regardless of us formalizing it and "discovering" the truth, and true whether or not there is any person thinking about it at any given moment. However, the Kantian argument makes a slightly weaker (but hopefully more defensible) argument than Frege makes. They diverge on the point of whether the Pythagorean theorem would be true if humans simply never existed at all. Kant commits himself to saying that this is indeterminate: much like Quine says that pragmatic considerations are exactly what evidence is, reality for Kant is exactly what it is possible for us to experience. For us to say anything at all about a world in which we do not exist would be a contradiction for Kant, as it would necessitate claims about *noumena* that are off limits to our senses. The risk associated with a Kantian position exploiting his world of possible experience is adopting all the rest of his machinery, including the forms of sensibility that are essential to his philosophy of mathematics. So far, all we have committed ourselves to from Kant's philosophy is the distinction between *phaenoumena* and *noumena*, which does not necessarily commit us to forms of intuition, and so we can consider ourselves, as of now, free of Kantian machinery.

4. *Reflected in Material World*

The theme of math objects being instantiated, reflected, manifested, participating in, or whichever specific formulation one might prefer, in the sensible world stands now less as a theme on its own. Rather, we can see it as necessary to mathematical realism in two ways: as a corollary of our adoption of poly-ism, or as an addendum to fulfill the applicability demand. In the first way, since we allow for a stable realm in our ontology precisely to organize our sensible experience, it follows that our sensible experiences reflects the stable realm. Since we take the stable realm to be comparatively ontologically fundamental, due to epistemological certainty, as already discussed, we take the sensible to be the reflection and manifestation, rather than the stable being a derivative of the sensible, which would contradict the second theme deemed essential to mathematical realism. In the second way, since any mathematical philosophy, as Shabel tells us, must answer two questions of applicability and apriority, mathematical realism must as well. In spite of the focus being on the ontologically prior, stable realm for the mathematical realist, we must still address the unreasonable effectiveness of mathematics, and it is easily addressed by adding to our philosophy that the sensible participates in the stable, and hence mathematical propositions from the stable realm structure the rules by which entities in the sensible realm interact.

5. *Changelessness*

Finally, we deal with the theme of mathematical propositions and objects being unchanging, which is related to mathematics being mind-independent, as seen in Frege's discussion of the Pythagorean theorem being independent of humanity and therefore changeless. What we must answer is, in light of our adopting a Kantian view of mathematical objects as being independent of specific humans discovering and doing mathematics, in what sense we must say mathematical objects are changeless, and whether it is necessary at all for us to say that they are changeless as mathematical realists.

Given what we have said to justify our stable realm in the poly-ism discussion, it seems that we must have them be changeless in some manner, but changeless with respect to what? We might say mathematical objects are changeless, regardless of any other factors; this would certainly be the strongest position, insofar as it posits the most, but it would not necessarily be the most defensible. For instance, as we have said, it is impossible for us to say anything about a world in which we never existed, and so we are not in a position to posit that mathematics will continue to be as it was in such a world. If we limit ourselves to what we are in a position to say, we might say that since math is the necessary condition for our experience to be structured as it is, math is changeless with respect to the machinery of our senses. It seems we must, then, adopt Kant's forms of sensibility to say mathematics is changeless with respect to these forms. If we do take the Kantian position, then the temporal changelessness Frege is referring to is a little less assured on a grander time scale: for instance, if the dinosaurs had different forms of sensibility, they would have a different mathematics (as different from our mathematics as their forms of sensibility were), and math would change with respect to time from the Paleolithic period to now. Hence, mathematics is changeless with respect to what makes it possible for us to experience the world, if we accept more Kantian machinery.

Having shown each theme to be necessary to a sufficiently defensible form of mathematical realism, we are now in a position to articulate such a position, and give ourselves reasons to prefer this position. I am personally convinced, and will argue later, that our first entry point into any philosophy is the world around us, more precisely the sensible realm, and that what we perceive is epstemologically prior to any other possible philosophy. Hence, as we have previously argued, the sensible world is full of contradictions: that which is cold becomes hot, night turns into day, pencils look broken when put in water but magically reform when taken out, among other such contradictions. This is the chaos of the senses, and due to our (by which I mean humanity's) conviction that the world ought not be contradictory, couched in our intuitive notion of the principle of non-contradiction derived from the sensible realm, we then discover within ourselves access to the stable realm that oganizes the perceptual realm.

It must be made clear that the stable realm does not only exist within each individual human's mind. The stable realm is a shared way by which humanity organizes our perceptions, and is only accessible through our rational faculty and application of the principle of non-contradiction to discover truths about the world, as far as it is possible for us to do so given the limitations of our senses. Within this stable realm exists mathematical objects that render true mathematical propositions that are formulated with bounded variable, and are consistent. These objects and propositions are reflected in the material world insofar as we discover this stable realm by way of looking at the relations of things or objects in the perceptual realm, and the stable realm organizes the raw data we get from the perceptual realm. This stable realm is also unchanging once it has been discovered. Our understanding of this realm might change, since we are not perfectly rational beings without exercising these rational faculties, but the mathematical objects are not themselves shifting even if we are to redefine a concept, as in our example of a limit, or formalize a system differently, as has also been discussed in our historical account of non-Euclidean geometries. Notice, just as we did when exploring the themes identified, our position borrows heavily from the Kantian philosophy of mathematics: more specifically, we do not attempt to deal with the *noumena* or anything outside the realm of possible experience which, as we will show in the following, is a reason to prefer our position.

Finally, we will give ourselves reasons to prefer this position, beyond the fact that our articulated position has the essential features of mathematical realism we identified previously. We will divide these reasons into two, not necessarily mutually exclusive, categories: reasons for a philosopher to prefer this position, and reasons for a mathematician to prefer this position.

First among the reasons to prefer this position for a philosopher is that the entry point is the perceptual realm, which is epistemologically prior. This is preferable since to do otherwise would be dishonest, both to the literature that favors a starting point in the perceptual, as well as to common sense and reason that philosophy is borne out of in the first place. There is value in starting a philosophy on grounds that are agreeable to as many interlocutors as possible, including those who may not be well-versed in the philosophical literature; the same sentiment is echoed in Plato's dialogues and Quine's essays, where they start from common sense principles that anyone, philosopher or not might agree with.

In addition, another virtue of the position is our adoption of the principle of non-contradiction, and our structuring of the stable realm in line with this principle in order to give it the consistency which our naïve understanding of the perceptual realm does not have. Moreover, rather than taking the principle of non-contradiction as an axiom, we attribute it to something essential to human nature: for instance, if you were to pull any Lawrence off of the road and ask whether someone could both sit and not sit at the same time (and same respect, same universe, and so on if you had a specifically curious Lawrence), he might say “if it weren’t for philosophers like you no one would think it was anything but impossible.” This is all to say, the principle of non-contradiction is part of the intuitions that we gain, and that there is sensory data to violate it is what motivates our search and subsequent discovery of the stable realm. As such, consistency is an important feature of the position, and is treated as such by giving it both epistemological priority insofar as it is intuitive to humans, and ontological priority insofar as it structures the ontologically fundamental, stable realm.

Another reason to prefer the position we articulated is that it meets the criteria we applied to the positions looked at previously, namely the apriority and applicability demands. Our position meets the apriority demand because truths about the world as we perceive it are rationally accessed, and do not depend upon the perceptual realm for their truth-values. The fact that there are without-a-doubt truths to be known about our world, in the absence of empirical testing or the scientific method, answers the question of how mathematics is deductively certain and consistent in and of itself. This apriority does not come at the expense of meeting the applicability demand, as the reverse does for the materialists: the unreasonable effectiveness of mathematics is accounted for by the fact that we are motivated to discover the stable realm through our observation of the perceptual world, and by the reflection of stable mathematical entities and relations in the perceptual world. For instance, that putting three more conservative judges on a Supreme Court bench gives you a 5-4 majority to overturn Roe v. Wade and revoke the bodily autonomy of half a nation’s population is a reflection of the proposition in the stable realm, two plus three equals five: specifically, five misogynists. Hence, our position also meets the applicability demand in a way similar to the many other poly-ist mathematical realists we have examined.

Finally amongst our reasons a philosopher ought to support our position, we do not make claims about that which is epistemological forbidden fruit, so to speak. Since we start from common sense, common ground as our philosophical entry point, we avoid positing that which we have no argument for knowing about, and abide by Occam's razor with respect to our epistemology, and therefore our ontology. Beyond the virtue of minimalist posits, this also means we avoid over-speaking or over-estimating our rational faculties: they are limited, perhaps less than our sensory faculties, but limited nonetheless. To act otherwise, as though our rational faculties can transcend the realm of possible existence that is hard-wired into us by virtue of the machinery of our senses, is to make our position both less defensible and less agreeable to the larger majority of people. I am not appealing to the masses, nor committing a bandwagon fallacy by listing this as an advantage; rather, I am claiming that humans are capable of recognizing our intuitions and by what way it is we know things. If we can appeal to the principle of non-contradiction through its widespread acceptance and intuitive nature, a similarly accepted and intuitive position, as long as it is valid and sound, ought to be similarly preferred by a philosopher who champions the principle of non-contradiction.

I claim there are reasons a mathematician ought to prefer our mathematical realism as well. First of these is that our position coheres with mathematical history, a point that is important to the development of the mathematical community and therefore the continuation of mathematics as a field. Just as mentioned, in reference to the historical development of the limit definition, while mathematicians change how they understand certain mathematical entities and revise their formalisms to better encapsulate this new understanding or create a consistent or more rigorous system, there is a notion of the correct concept we hope to capture by our definition. This correct concept is exactly the mathematical entity that exists in the stable realm. Hence, our position that holds there are stable, changeless (with respect to species, at least) mathematical entities, which coheres with the way in which mathematicians develop definitions for these stable entities.

Our meeting applicability demand also reflects, pun unintended, the way in which mathematicians model the real world through applied mathematics. Just as viewing the contradictions in the perceptual world stimulates us to look towards the rationalistic, stable realm, looking towards the endless barrage of empirical data gathered about a situation stimulates an applied mathematician to organize this data by way of stable, mathematical entities. Not only does the stable realm help organize this data, it can predict situations if the stable realm is well-employed and the mathematician creates a good enough model and give us new understandings of relationship of entities in the perceptual world. Hence, our view is compatible with the way in which mathematicians practice applications.

Finally, our adoption of the principle of non-contradiction is in line with the way in which mathematicians adopt axioms when creating mathematical formalisms. For instance, mathematicians accept that there are no integers between 0 and 1 as an axiom for the ring of integers on the basis of an intuitive understanding, which in turn is based on a desire to model something specific in the stable realm. Similarly, we accept the principle of non-contradiction on the basis of an intuitive understanding, which in turn is based on our rational, hardwired, stable realm. Hence, the way in which mathematicians practice, in this way, also corroborates our position on the ontology of mathematical objects.

Chapter 4: The Twelve-Step Realist Recovery Program

The goal of this chapter is to undo all the progress we have made so far, and convince an interlocutor, having taken the position we articulated in the previous chapter, that this position is untenable. By extension, I hope to cast more doubt in the minds of one who might be sympathetic to mathematical realism and caution against its adoption as a philosophy as a whole, in light of current mathematics and philosophical inconsistencies. Having done so, I will then also attack the Indispensability thesis, as articulated by Quine, in order to demonstrate that positive ontological commitments to mathematical objects are fallacious, regardless of whether or not they are substantive claims, that is, they posit mathematical objects as substances. Finally, I will explain why the reasons I gave to prefer the position articulated in Chapter 3 are either preserved by not taking a mathematical realist position, better served by an anti-realist position, or are simply not important enough considerations for us to prefer mathematical realism.

A charitable entry into any philosophical discussion is what points upon which we agree. So, the points from Chapter 3 that we agree to are that our entry point ought to be the perceptual realm, that raw reality unfiltered by our senses is epistemological forbidden fruit, that something must be done about the contradictions in the perceptual realm and organize our perceptions, and finally, we affirm the principle of non-contradiction as an intuitive epistemological fact. As mentioned in Chapter 3, there is no way we can argue at all without agreeing on what a valid argument even looks like, so to really enter this argument, we must accept the principle of non-contradiction as the basis of our shared logic.

Then, of course, we must state where we start to diverge. While we accept the principle of non-contradiction as an epistemological fact, we do not accept it as an ontological principle. Our interlocutor from Chapter 3 might say that then we fall into solipsism: since we do not say anything at all, having left ourselves in a position fraught with empirical skepticism and perceptual contradictions, we are vulnerable to an empty ontology that reeks of hopelessness. There could be an argument as to whether solipsism and hopelessness is truly a bad enough position for us to dismiss it entirely, and posit things just so we could avoid it. But I will leave such an argument to an existentialist and take the mathematical realist seriously in their fear of solipsism. I posit that, in addition to the realist's criterion of epistemological certainty necessary to add something to our ontology, we might also add that an object should have a perceptual basis for us to add it to our ontology. This would be preferable to the epistemological criterion without the perceptual pre-requisite, if we continue to take Occam's razor into account, as we affirm the existence of less, with the same consequences: that is, we do not end up in solipsism.

The weight of this phrase, "perceptual basis," ought to be talked about in some detail. By this, we mean direct sensory data, that is, we claim the existence of that we claim to actually sense. For instance, if we do not directly sense the gravitational force between the Earth and the Sun, we do not posit its existence, in spite of how epistemologically certain it might be. However, if we can directly perceive the sensory data we group under the term pencil, we will posit the existence of a pencil; or, more comparable to our previous example, if cells can be directly perceived under a light microscope, then we might adopt cells into our ontology. Notice, though, regardless of the technological advancements that allow us to perceive more things, such as the invention of light microscopes allowing us to perceive cells, mathematical objects will never be admitted into our ontology after the adoption of the perceptual criterion: there is no technological advancement conceivable to allow us to directly perceive mathematical objects, even if our ontology of scientific entities expands.

With respect to how it is that perceptions can be a good basis for an ontology at all, especially considering we are skeptical with respect to empirical data, this is why we combine it with the condition for epistemological certainty. For instance, say that Lawrence has taken Drug S, and claims to have rationally apprehended the idea of infinity. We reject infinity, including Lawrence's notion of infinity, from our ontology, due to our perceptual condition; Lawrence does not claim that he *saw* infinity, but apprehended it *a priori*, and while we might give it some epistemological weight, it will not have a positive ontological status. On the other hand, if Lawrence were to try Drug L and then claim that he saw a ghost in his room, we reject ghosts from our ontology not because we do not think Lawrence's sense data appeared to him in a certain way since we are not prescriptivists in such a way, but rather because it has no epistemological certainty since it is not accepted into our collective conceptual scheme, of the type we described as a stable realm in Chapter 3.

If we would rather not take Occam's razor as a reason to prefer our position, we can instead look towards how mathematics works: that is, our epistemological understanding of mathematics, as current mathematicians practice it, so that we can check whether it does actually meet the mathematical realist's criterion of epistemological certainty. Since current mathematics is axiomatized, even for propositions that are not conditionals, there are unsaid antecedents that show up in the proof, namely the axioms. A distinction between axioms and simply intuitive facts, as we class the principle of non-contradiction, is important in order for us to deal with, for instance, Euclidean and non-Euclidean geometries, but more generally with appropriately ascribing the logical relations, that is, what leads to what. Some axioms might be intuitive, epistemological facts, such as the principle of non-contradiction is an axiom for classical logic. Hence, propositions are certain *given certain axioms*: for example, the statement "for any line *l* and a point not on the line *P*, there is exactly one straight line passing through *P* that is parallel to *l*" is true given Euclid's five postulates. So, even this truth is conditional upon existing within a certain structure, rather than an absolute truth. Notice, this is all an epistemological move: we do not have epistemological certainty of mathematical propositions, since they are only certain given various axioms, and so we cannot apply the mathematical realist's criterion and adopt them into our ontology.

Our realist, pinned as such, might agree and recast the above statement in sentential logic, let us call it *T* for theorem, as follows: "$(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow T$" where each P_i is one of the postulates of the axiomatic system. Then, it is a true-in-all-cases proposition that has the epistemological certainty necessary for us to adopt it into our ontology. However, we can also play this game and recast this statement using predicate logic as "$(\forall x_1)(\forall x_2)\ldots(\forall x_m)[P_1(x_1, x_2, \ldots, x_m) \wedge P_2(x_1, x_2, \ldots, x_m) \wedge \ldots \wedge P_n(x_1, x_2, \ldots, x_m)] \rightarrow T(x_1, x_2, \ldots, x_m)$," where each x_i is an object in our axiomatic system, and *T* might be a proposition with either a universal or existential quantifier. The attentive reader might see where we are going with this, namely Quine's criterion for ontology, where the values of bound variables are adopted into our ontology.

If *T* is strictly comprised of universal quantifiers, the mathematical realist might accept that they cannot affirm the reality of the objects involved. Quine's argument comes into play if there is some existential quantifier in *T*: for instance, as in the theorem of Euclidean geometry previously stated, the mathematical realist could invoke Quine's argument in order to say that the mathematical object "line passing through *P* and parallel to *l*" is a positive ontological commitment of mathematics that we must accept since what we take to be certain says it exists. However, the mathematical realist ignores the universal quantifiers that prefix the axioms: and these largely must be universal quantifiers, as they tell us how objects interact within our axiomatic system[16]. It is only by affirming the truth of the antecedent do we have certainty of the consequent (via a Modus Ponens). However, the certainty of the antecedent is either vacuously true when none of the mathematical objects exist, or cannot be shown to be true without begging the question on the mathematical realist's end. Hence, the mathematical realist cannot apply their criterion, or the Quinian criterion, in order to affirm the reality of any mathematical objects.

This is not the only objection I pose to mathematical realism: one might balk at my previous objection because of the overly technical nature that recasts statements in order to attack the realist, and take it as uncharitable, logician jargon. Rather I will consider a case that we did not treat properly in formulating the mathematical realist's position: Euclidean and non-Euclidean geometries. Let us take the realist at their word, use the criterion of epistemological certainty as evidence of a proposition's ontological truth value, and assume that this entails a positive ontological commitment to the objects of this proposition, ignoring our talk of bound variables. Then, since both Euclidean and non-Euclidean geometries are consistent, that is, they align with the principle of non-contradiction, and one is just as deductively sound as the other, both are epistemologically certain. Hence, we ought to grant the propositions and objects involved an ontological standing, both from Euclidean and non-Euclidean geometries: so the

[16] I qualify this with "largely," since there are axioms that include existential claims of the type ensuring the existence of an identity element. For example, the empty set is guaranteed existence in the Zermelo-Frankel axioms for set theory, and an identity element in ensured in the axioms of group theory. This does not provide a counterexample to my argument, though. One axiom with an existential quantifier does not outweigh the rest of the axioms being universal quantifers. For the sake of argument, however, assume an axiomatic system in which all the postulate are existential statements: the axioms still are antecedents, and only by showing that each of the objects mentioned in the axioms exists would the theorem be proven, hence beginning the question on the part of the mathematical realist. In addition, it would also not be a very good axiomatic system, with limited explanatory power, nor a very interesting system, as none of the theorems would be powerful enough to say something substantive, as a universal proposition cannot be derived strictly from existential statements.

unique line parallel to *l* passing through *P* exists, as does the multiple lines parallel to *l* passing through *P,* and the proposition "there are no lines parallel to *l* passing through *P*" is also true. If we are to recast these propositions with predicate logic, as we did before, there would be no contradiction: but we would also not be able to posit the witnesses of our existential quantifiers without wading through the universal quantifiers in the antecedent, working against the realist. Ignoring the axiomatization and quantification, then, multiple contradictory statements are true at once for the mathematical realist, contradicting the idea that reality abides by the principle of non-contradiction.

However, the earnest realist would say that we should not accept contradictory statements epistemologically too. Even as we deal with them before giving them ontological reality, we recognize that the propositions contradict each other and violate the epistemological fact of the principle of non-contradiction. Hence, we do not accept both into our ontology and do not fall into a contradiction in reality. This does not answer the question of which geometry we do adopt: either we adopt none, in which the anti-realist has won on this count; we adopt them all, and we end up with a contradiction in our epistemology and ontology, or we adopt only one of them, and must have a good reason that this one is the only one we accept that is consistent with our criterion. So, what could possibly be our bright line? The earnest realist might want to take Euclidean geometry as real over the other two possible geometries, but that could be for historical reasons or because it is intuitive and best organizes our sensory data. In the first case, historical reasons are not sufficient as an *a priori* reason to accept one thing into our ontology over another: just because it has been lauded more or was "discovered" first. In the second case, the realist adopts our extra criterion of that which has sensible evidence to back it up, but a weaker version in the sense that we require for it to actually be in our sensory data, whereas the realist will accept its reflection to be present in sensory data. For instance, I would not accept a perfect circle into my ontology, because there are no perfect circles in my sensory data, but a mathematical realist would accept a perfect circle into their ontology because it is both epistemologically certain and a reflection of it is present in sensory data.

This modification presents two problems for the mathematical realist: it does not solve the problem of deciding between geometries and does not allow them to affirm mathematical entities that do not give them problems but are more abstract. The first problem is this modification is an insufficient bright line to decide whether to affirm the reality of Euclidean or non-Euclidean geometries. While we might see the reflections of Euclidean geometry more in our daily life, it is not always the case that our sensory data supports Euclidean geometry. For instance, outer space is best modeled with, and reflects hyperbolic geometry. Thus, this extra sensory criterion ends up being too weak to do what the mathematical realist might want. The second problem is that in requiring a sensory reflection in order to affirm its ontological standing, the mathematical realist excludes more abstract mathematics from their ontology that they might not necessarily care to exclude. For instance, point-set topology is an axiomatic system concerned with open sets as its objects, which has no reflection in our sensory data: the primary example of an open set used to give topologists an intuition of the field is an open interval in the real numbers, which is also abstract and without evidential sensory data.

On the flip side, a mathematical realist would want more abstract mathematics to have ontological standing, as it underpins other useful ideas. Topology is a more general version of Real Analysis, which is the foundation of Calculus, the typical example of mathematics being unreasonably effective. A good realist would want to give Calculus ontological standing, requiring them to give Real Analysis ontological standing in order to meet the consistency criterion. Having accepted Real Analysis, they would also want to accept Real Analysis if the idea of an open set was a bit more general, that is, they ought to want to give topological objects ontological standing, which they cannot with the sensory criterion. Hence, the criterion is also too strict for them to ontologically affirm all mathematical systems that do not contradict others, including those they might want to give reality to, such as Topology. Given that adopting a sensory criterion gives the mathematical realist two problems, and does not resolve the one they hoped to by adopting it, and without any criterion they end up in problems previously mentioned, we consider this to be a strong argument against realism.

These contradictions and undesirable consequences form the basis of our anti-realist position. As we agreed with the realist, the appropriate entry into this discussion is our sensory data, which has contradictions that make us skeptical of the perceptual realm. Rather than positing a stable realm, however, in our view we resolve the contradictions through the purely epistemological frameworks of mathematics and science that do not have ontological standing, but rather have heuristic utility: neither molecules nor sets exist, but they are useful tools to organize our sensory data. Then, the question of the unreasonable effectiveness of mathematics is answered exactly by the way in which it is constructed. We construct our mathematics for the very purposes of organizing our sensory data without contradictions, and math simply fulfills its function well: sometimes it requires changes or vast structural change, as in the formalization of Calculus to rigorously define a limit to better model how heat transfer works, and this is further evidence for our position. Since math is only epistemological, math really is changing as our understanding changes and as we revise our definitions to better organize our sensory perceptions or align with the principle of non-contradiction.

By not positing a stable realm, the mathematical realist balks at the idea of not being able to make necessarily necessary claims. However, it is not necessary for us, pun unintended, to do so to meet the apriority claim and have a valid philosophy of math according to our previously stated criteria, having already demonstrated how we meet the applicability criterion. Mathematics is still *a priori* since it does not depend entirely upon sensory data to construct it, or else it would not be able to resolve the contradictions of perception. Rather, mathematics is constructed in the mind as an idealized, consistent version of the perceptual realm, and becomes more and more abstract as mathematics as a field develops, allowing for structures such as Topology to exist as a result of human curiosity and other historical factors. As such, our position meets the apriority demand, as mathematical structures are not bound to sensory data: the only thing these structures are bound to is the principle of non-contradiction (at least, for classical mathematics, but that is a separate discussion), which is an *a priori* epistemological fact. Hence, we manage to meet the apriority demand just as well as the realist, without needing a stable realm and without the need to posit more than absolutely necessary and meet the applicability demand better than the realist.

Another possible objection to our anti-realist position might be the issue of the distinction between the category of mental entities that unicorns fall under and the category of mental entities that mathematical objects fall under. Under our position, it seems there is no such distinction, which is correct. Unicorns come about by looking at sensory data, such as horses and animals with horns like narwhals, and then combining them into a single mental entity: this is a method of abstraction by which two ideas are synthesized into a single one to form something new. Similarly, mathematical entities are constructed by abstracting from sensory data, or abstracting from other mathematical entities: for instance, sets are an abstraction of grouping things together in the sensory world to create a new mental entity, and much of the formalisms in math are abstractions from set theory. The realist objects to this because mathematical objects obey rules and are more strictly defined than unicorns. Once more, this misconstrues how mathematics following the turn of the century works: the rules that the number 2 follows is determined by the structure it is within, just as the magical laws a unicorn obeys is determined by the fictional universe it is set within.

There are reasons for both the philosopher and the mathematician to prefer our position, specifically over the realist position articulated in Chapter 3. As far as the philosophers are concerned, we continue with some of the benefits mentioned in the last chapter. For one, we continue to give epistemological priority where it is due, to our sensory data, and we also continue to treat reality unfiltered by our senses as epistemological forbidden fruit. In fact, we do better in recognizing that reality *sans* senses is not anything it is possible for us to make claims about at all: while the mathematical realist tries to apply to principle of non-contradiction to reality *sans* senses, we do no such thing in the anti-realist position. Mathematics stays exactly where it ought to, simply as an aid in organizing sensory data, without us needing all the ontological baggage nor needing to posit anything outside of the realm of possible experience, as Kantian as it might sound.

In addition, our position also affirms the principle of non-contradiction as an epistemological fact, and we are able to get absolute truth without extending the principle beyond our possible experience. As mentioned in our tangent on the matter of quantifiers, we can imagine any theorem within a mathematical system to be the consequent, and the axioms of the system to be the antecedent (with largely universal quantifiers in front of these axioms). Since proofs of mathematical theorems proceed deductively, the entire proposition with postulates implying the theorem is, indeed, absolutely true: that is, it is as true as we can say anything to be, and exactly follows the recursive definition of truth in a logical system. Propositions still have truth values, then, but this truth value is simply dependent upon the system, which is consistent with what was said earlier: the way that mathematical objects "act" is dependent upon the system it is within, and we define these conditions in how we construct our axiomatic systems[17].

A mathematician ought to prefer our position because it best models their actual practice, that is, with axiomatic systems. Of course, this is a controversial statement: many philosophers claim that mathematicians act as Platonists when they are actually doing math, and even a mathematician might say they are "a Platonist until 5 PM on Friday each week." This does not mean that every mathematician is in fact a Platonist, and any other philosophy of math disrespects mathematicians as experts in the field they study. While mathematicians might have the methodology of Platonists, their philosophy need not be as such: there is an important distinction to be made between how one does something, and their philosophy regarding it. Methodology does not imply philosophy, nor does philosophy imply methodology. For instance, many contemporary logicians study how a logical system acts without the principle of non-contradiction, such as intuitionistic logic, which requires constructivist proofs to prove a theorem valid and true. This does not imply that the logicians studying this do not "believe" in the principle of non-contradiction: there are no judgments we can make on a mathematician

[17] Insofar as I have spent this entire thesis talking about, and then arguing against, mathematical realism, this might beg the question of what my actual position is. From my construction of the anti-realist position, one might try to attribute an intuitionist position to me (best articulated by Henri Poinaré in *Science and Hypothesis*). This is a coincidence of the positions we saw in Chapters 1 and 2, which determined the position of Chapter 3, which in turn determined what arguments and stances I take in this Chapter. If the historical trend of mathematical realism was more concerned with mathematics as deductively certain, the positions we may have surveyed might lead me to take a formalist approach (previously mentioned in Chapter 2, and attributed to David Hilbert) as a more preferable position without ontological baggage. One position I would ascribe to myself, however, is structuralism (articulated by Paul Benacceraf in "What Numbers Could Not Be"), which is the influence for my claims about the interaction of mathematical objects being determined by the axiomatic system which they exist within.

based on what they study[18]. Our position reflects exactly that, by not ascribing any philosophy to any methodology. Rather, our counter-position simply claims that current mathematics does not necessarily lead us to accept realism, and we can have all the consequences of realism, most importantly organizing our perceptions, without needing any extra ontological commitments.

Another interlocutor might, while staying within the bounds of this thesis, not accept the mathematical realism that we put forth, but still claim that mathematical objects exist, if they are a Quinian. So, we must give this interlocutor good reasons to not prefer the Quinian position, specifically for rejecting the indispensability thesis. The indispensability thesis consists of a criterion for adopting something into the ontology and then claiming that this criterion applies to mathematical entities, in order to conclude that we ought to adopt mathematical entities into our ontology. Hence, there are two claims that Quine makes that we must grapple with: that his is a good criterion to use in order to adopt things into our ontology, and that this criteria apply to mathematical entities.

His criterion for adopting something into his ontology is that it is indispensable to our best scientific theories: that is, no more and no less than what we need to make our best scientific theories work is what he ontologically commits himself towards. The ontological criteria we posit in our anti-realist position are two-fold: that the propositions we adopt are epistemologically certain and aligned with the principle of non-contradiction; and that the objects involved in the propositions are supported by sensory data. Quine might argue that our criteria bottom out in the same thing: our best scientific theories tend to be consistent, since the principle of non-contradiction is central to the web of belief, and our best scientific theories are given such

[18] This is not entirely true: there is (at least anecdotal) evidence to suggest that algebraists eat corn in rows, whereas analysts eat corn in spiral. At the very least, then, by a mathematician's research we might predict how they eat corn.

status precisely because they are best supported by our sensory data. This is not the case: scientific theories *tend* to be consistent, but it is certainly conceivable in Quine's view for them to no longer be consistent, if the principle of non-contradiction is unseated from its privileged position at the center of our web of beliefs. This is not a problem that our position has. Since the principle of non-contradiction is explicitly stated as part of our criteria, there are no entities adopted into our ontology that might introduce a contradiction: a pencil that is straight is not accepted into our ontology along with the same pencil at the same time, in the same place, and same respect, that is bent.

If we are to set aside concerns about contradictions, though, our positions still do not end up doing the same thing. Quine might claim that the force of gravity and Newton's gravitational theory do not introduce contradictions, and it has sensory support since when things are thrown up, they eventually come down, exactly as our theory of gravity predicts. Hence, we anti-realists ought to accept gravity into our ontology, at the very least, and since talking about gravity requires the gravitational constant, perhaps we ought to accept that as well into our ontology until Quine gets us to accept sets are real. Of course, we do not want that. So, we draw a distinction between direct sensory evidence and indirect sensory evidence. Direct sensory evidence is of the type we mentioned when adopting pencils or cells into our ontology: what we see is the pencil or cell through a microscope, and we do not deduce anything else from this sensory data. What Quine would be using to try and get us to accept gravity into our ontology would be indirect sensory data, that is, seeing an amalgamation of sensory data, observing a pattern (things that go up come back down), and then deducing from this sensory data that there must be something to explain these phenomena that we cannot directly perceive, the gravitational force of the planet. These are very different processes: the pencil is directly perceived by our senses while the gravitational force of the planet is not and only explains sensory data. Hence, our criteria still do not bottom out in the same way.

In addition, our criteria is preferable to Quine's indispensability criterion since we end up with stable truth values, which Quine cannot guarantee. Since truth is always revisable in Quine's system, there is no proposition for which he can guarantee the truth value: earlier we demonstrated how he cannot even guarantee the truth value of the principle of non-contradiction. In our position, the truth value of a mathematical proposition might change: for instance, as we rigorize our understanding of a limit, we might shift from saying that the limit of a series is 0, or 1/12, and instead saying it does not exist. However, keep in mind the recasting with quantifiers we used earlier in this chapter. It would be just as easy for us to add the definition into the antecedent of our proposition as yet another hidden postulate of sorts, and so it is a different proposition we are affirming to be true, with a different antecedent: the previous proposition was true with the previous definition, and the current proposition is true with the current definition. This is yet another reason we are affirming structuralism as the proper way to view mathematics: propositions have their truth value within a specific structure.

Quine's position only worsens if we are to draw the distinction between epistemological truth values and ontological truth values. For Quine, ontological truth values are subject to change: the statement "the four humors exist" has a truth value that is unstable, so something can exist, and we might later say it never existed. Just as easily, we might later say that gravity does not exist. This is a change in his ontology, giving him a state of reality that is unstable, defeating the very purposes we made clear in Chapter 3, that is, to have a stable realm in which to determine truth values. Hence, even as a realist, his position is not preferable. In contrast, our position only goes so far as to talk about epistemological truth values: if these are unstable, the only thing that changes is our understanding of things, not whether or not they have ontological status any longer. An interlocutor might object that epistemological certainty is a criterion for adopting something into our ontology, so if epistemological truth values can change, so can ontological truth values in our position. However, our other criterion is being confirmed in sensory data, so abstract entities are never admitted into our ontology. The only way the objects we commit ourselves to ontologically could change is if the way we experience the world as a species radically changes in order to alter our sense data, or if all of the sensory data that we identified as a certain object were to disappear. For example, there are two ways in our ontology that we might decide to say pencils do not exist: either all humans develop a condition called pencil blindness, all at once, making them completely unable to perceive pencils, or all pencils were to disappear from the earth entirely. Neither case is very likely, and in order for them to become true objections to our position, everyone would also have to forget that pencils *ever* existed, as otherwise the propositions might be amended to have a temporal element. Whatever the case, our position is still preferable to Quine's in that we have stable truth values that, by necessity, do not end up in contradictions.

The second set of objections we can levy against the indispensability thesis is in applying it to mathematics. What we claim is that it is certainly conceivable for science to exist separated from mathematics, and in fact, science might be better off without its many appeals to mathematics. The claim is often that mathematics makes our measurements more precise, but this is not a very heavy use of mathematics: carpentry similarly uses mathematics for measurements and recording, but we do not see this as good evidence for numbers being real. Mathematics as used for measurement is hardly mathematical and does not quite demonstrate whether mathematics is unreasonably effective as indispensability is trying to get at. So, if we leave precision by the wayside, the indispensability of mathematics must be referencing the predictive power of mathematics joined with science. For instance, Quine could claim that it would have been impossible, or at least infeasibly difficult, for us to get to the moon without using mathematics.

However, with all the mathematics used at NASA, multiple space shuttles have still exploded, resulting in lives and money lost. These explosions did not happen because the mathematics did not work out as it should, but because it was not applied well. There were hidden factors at play, such as the humidity on the day of launch, the temperature the parts were stored at, or one of a million other factors that might lead to such a tragedy, none of which would be captured by the mathematics. The issues are only solvable by repeated applications of the scientific method, testing in multiple conditions, and trying to account for as many factors as possible. Similarly, the reason we were able to get to the moon was because of the multiple tests of equipment, learning from previous failures, and adjustments to the process to account for these newly discovered relevant factors. Mathematics was only a framework that could be applied to all the scientific data: it was a way of organizing the data from multiple experiments that certainly made it easier to get to the moon, but was not indispensable. The fundamental problem of science is usually the inability to account for all possible factors like an omniscient being might. This is a problem that Descartes is concerned with as well, in his talk about the asymptotic matching of discrete states of being to treat them as though they are continuous, such as in modeling the inelastic collision of billiard balls. Mathematics does not solve this fundamental problem, only helps organize what we find in attempting to solve it. Hence, mathematics was not indispensable to our ability to go to the moon: to be pedantic about it, the insatiably curious human spirit that leads us towards science was indispensable.

Hence, we claim that mathematics is not indispensable to science, but rather is only a framework for organizing scientific results. Mathematics is not applicable without the use of the discipline it is being applied to: modeling a menstrual cycle is not possible without medical knowledge of hormonal cycles and uterine lining, but it is certainly possible for a gynecologist to say something that a biomathematician might find from their model, simply by seeing multiple cases that tell them what affects the menstrual cycle. Mathematics and science are certainly separable and go about looking for truths in the empirical world in different ways: one through testing, and one through predictive models. However, the mathematician still requires some data to base their model on, or some scientific findings to justify the relationships they model in their equations. The mathematician cannot talk about empirical truths without the scientist (or practitioner of whatever type), but the scientist can talk about empirical truths without the mathematicians but might borrow from the mathematicians in order to make what they are saying more epistemologically certain. Therefore, we hold that mathematics is pragmatically useful for our best scientific theories, but is not indispensable for science to progress.

For these reasons then, both that Quine's criterion is not preferable to our criteria for what we ought to adopt into our ontology and that Quine's criterion does not apply to mathematics or justify that it is indispensable to science, we hold that Quine's position is not preferable to our own position. The upshot of this is that, in not preferring Quine's position, combined with our previous arguments against mathematical realism, we have presented in this chapter reasons not to have ontological commitments to mathematical entities, which as we stated as the top, was exactly our goal.

Conclusion

I will absolutely not claim that my position is the end-all, be-all of mathematical anti-realism, and there may be multiple objections to levy against this paper. However, the value of philosophy is not in solving problems but in generating interesting discussions around the question of the reality of mathematical objects. Hence, while I may have had certain goals with respect to what philosophers I wanted to discuss and which arguments I wanted to put forward into this long, long history of the philosophy of mathematics, my real goal was to foster a long-running philosophical discussion on mathematical realism among my committee and others reading this thesis, which I believe to be the real goal of philosophy: encouraging curiosity as to why things are the way that they are. If I have given any reader of my thesis the tools to continue asking philosophical questions about mathematics, given them an excuse to seriously engage with these questions, or simply sparked any kind of academic interest in the questions we have discussed in this thesis, I will take that as a personal success.

With the sentiment aside, there is obviously plenty of future directions this project could go in, either having examined the parallel or off-shoot development of other schools of thought in the philosophy of mathematics, such as intuitionism or formalism, constructed an orthodox and most defensible version of this position, and then used it to lobby both negative and positive attacks against the realist position I have examined here. Another avenue might be a broader treatment of mathematical philosophers after Quine, which gets even more technical than the values of bound variables much of the time. The technicalities have a lot of their own value, as they must, being an entire field of study in both mathematics and philosophy, and I hope to have given a taste as to why the logical technicalities are important for discussions in the philosophy of math.

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