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Cayley Map Embeddings of Complete Graphs

Miriam Scheinblum
mscheinblum@rollins.edu

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Miriam Scheinblum

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MIRIAM SCHEINBLUM

1. Abstract

This paper looks at Cayley map embeddings of complete graphs on orientable surfaces. Cayley maps constrain graph embeddings to those with cyclical edge rotations, so optimal embeddings on surfaces with the minimum genus may not always be possible. We explore instances when Cayley maps succeed at optimally embedding complete graphs, and when optimal embeddings are not possible, we determine how close to optimal they can get by finding vertex rotations that result in the smallest possible genus. Many of the complete graphs we consider have prime numbers of vertices, so for each complete graph K_n we focus on mappings with the finite cyclic group Z_n.

2. Introduction

The topological study of graph embeddings dates back to proofs of the Euler equation in the 1800s [6]. Perhaps the most notable work relating to graph embeddings on surfaces is that of Gerhard Ringel, whose proofs of the Heawood map-coloring problem involved rigorous calculations of the genera of optimal embeddings of certain types of graphs [8, 9]. Ringel's work helped pique interest among other mathematicians regarding optimal graph embeddings in the 1960s and 1970s [6]. Embedding a graph is essentially drawing the graph on a surface without allowing any edges to cross each other. For the purposes of this paper, think of a surface as the two-dimensional representation of the outer layer of a three-dimensional solid, such as a ball or donut.

Graph embeddings have applications outside of just the field of topological graph theory, for instance in the design of printed circuit boards [5]. Electrical circuits can be modeled by graphs, where edges between vertices are the wires between connection points. To avoid short circuiting, wires cannot cross in electrical circuits, just as no edges can cross in graph embeddings. Since it may not be possible to connect all the wires without any crossings on a flat chip, printed circuit boards often have holes and several layers that wires can be routed through to avoid edge crossings. The addition of layers is comparable to the addition of holes in orientable surfaces of increasing genera. Graph embedding problems can help determine how to best design and construct these circuits.

As you can imagine, coming up with large, optimal graph embeddings can be complex and time consuming. Cayley maps simplify this process by symmetrically embedding graphs on orientable surfaces. This process is not always guaranteed to achieve the optimal embedding, but it makes determining and representing proper graph embeddings much more efficient than other methods. In this paper, we are interested in exploring how close to the optimal genus a Cayley map embedding can get. We choose to study the embeddings of complete graphs in particular since these complete graphs contain all other simple graphs within them. We break down our study of optimal Cayley map embeddings of complete graphs into several sections.

We will first define terminology and introduce notation in Section 3 to provide the necessary framework for understanding Cayley map embeddings. In section 4, we will look at the optimal embeddings of some small complete graphs. Then Section 5 introduces and proves lemmas that will be essential in proving that some larger Cayley map embeddings are as good as we can get. The lemmas help us with K_{11} in Section 6, K_{13} in Section 7, and K_{17} in Section 8. Section 9 lists optimal Cayley map embeddings of complete graphs of the form K_{12m+7} for nonnegative integers m and raises a conjecture about embeddings for all $m \geq 0$. Some bounds for the number of faces and genera of embeddings are generalized in Section 10. Finally, we have three appendices. Section 11 summarizes and compares the achieved genus and optimal genus of each complete graph discussed in the earlier sections. A few supplemental lemmas are provided in Section 12. Finally, Section 13 lists the full Python code used in support of the Section 9 conjecture.

3. Definitions and Notation

This section provides a brief introduction to various concepts from graph theory, group theory, and topology. A basic understanding of these concepts will be necessary for grasping the nature of our work with Cayley map embeddings. Additionally, we will define important notation that will be used throughout the paper. First, we will start by introducing the idea of a graph.

A graph $G = (V, E)$ is made up of a set V of vertices (i.e., points) that are connected by a set E of edges (i.e., lines). Graphs can be classified based on how their vertices are connected by these edges. We will be exclusively looking at complete graphs.

Definition 3.1. *A complete graph K_n has n vertices and an edge between every pair of vertices, for a total of $\frac{n(n-1)}{2}$ edges.*

When drawing graphs, we often want to minimize the number of edges that cross over each other. Sometimes it is possible to eliminate all edge crossings when drawing a graph on a flat plane (in which case the graph is planar), but oftentimes we must draw graphs on other surfaces to achieve this desired result. We do this by embedding graphs on orientable surfaces with increasing numbers of holes, such as the sphere, torus, double torus, three-holed torus, and so on. The more holes a surface has, the more potential routes an edge can take to connect vertices without crossing other edges on the way.

In 1866, Jordan showed that, up to homeomorphism, the set of closed, orientable surfaces consists of the sphere, torus, double torus, etc., which have genera 0, 1, 2, etc., respectively [7]. An embedding of a graph $G = (V, E)$ maps the vertices V and edges E onto a surface in such a way that no edges cross.

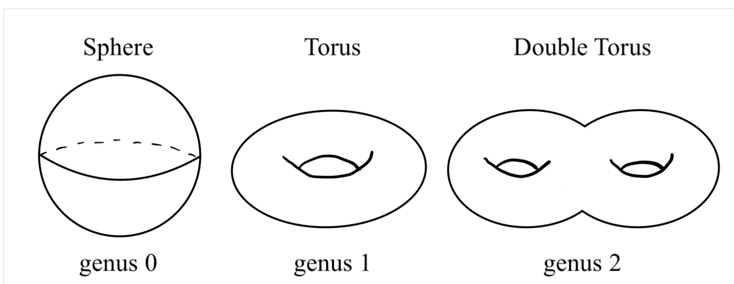


FIGURE 1. Some simple orientable surfaces

Definition 3.2. *An embedding of a graph $G = (V, E)$ onto a surface S consists of*

(1) *a one-to-one function $f_V : V \to S$; and*

(2) *a continuous, one-to-one function $f_e : [0, 1] \to S$ for each edge $e \in E$, such that if e connects vertices v_0 and v_1, then $f_e(0) = v_0$ and $f_e(1) = v_1$ (or $f_e(0) = v_1$ and $f_e(1) = v_0$)*

with the property that $f_{e_1}(x) = f_{e_2}(y)$ for any $x, y \in (0, 1)$ implies $e_1 = e_2$ (and $x = y$).

The set of paths corresponding to the images of the functions f_e for all $e \in E$ divide the surface into components. Each component is a face of the embedding. If every face is homeomorphic to an open disk in $\mathbb{R}^2$, then it is a well-known fact from algebraic topology that the Euler characteristic

$\chi = |V| - |E| + |F|$, where $|V|$, $|E|$, and $|F|$ are the number of vertices, edges, and faces respectively, is a surface invariant. The genus g of an orientable surface is determined by $\chi = 2 - 2g$.

An optimal embedding of a graph G embeds the graph on the surface with the smallest genus, which we denote as $\gamma(G)$. The genus of an optimal embedding of a complete graph K_n is given by Ringel and Young in Theorem 3.3 below [9].

Theorem 3.3. *The complete graph K_n has optimal genus*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

Given an embedding of a complete graph K_n, genus $g \geq \gamma(K_n)$ since the embedding is not guaranteed to be optimal.

Definition 3.4. *A group G is a set with:*

(1) *An associative binary operation $*$ (i.e., for all $a, b \in G$, $a * b \in G$);*
(2) *An identity element e (i.e., for all $a \in G$, $a * e = e * a = a$);*
(3) *And an inverse for each element (i.e., all $a \in G$ have an inverse $c \in G$ such that $a * c = c * a = e$).*

Definition 3.5. *A group G is abelian if the operation $*$ is also commutative (i.e., for all $a, b \in G$, $a * b = b * a$).*

The order of a group element x, denoted $ord(x)$, is the smallest positive integer m such that $\underbrace{x * x * \cdots * x}_{m \text{ times}} = e$. Depending on the group operation $*$, people primarily use either addition notation (e.g., $x + y$) or multiplication notation (e.g., $x \cdot y$) when working with group elements. Thus, we define $ord(x)$ more specifically as the smallest positive integer m such that $mx = e$ or $x^m = e$ using addition or multiplication notation, respectively.

Additionally, the inverse of a group element is denoted differently depending on whether addition or multiplication notation is being used: the additive inverse of x is denoted $-x$ and the multiplicative inverse of x is denoted x^{-1}.

An example of an operation that uses addition notation is modular arithmetic, a method of counting that restricts all possible numbers to a finite set of integers, cycling back to the first number once the largest number is reached.

Definition 3.6. *Addition modulo n uses the integers $0, \ldots, n-1$. Given an integer a, we define $a \bmod n = r$, where r is the remainder upon dividing a by n.*

In this paper, we will be using a type of abelian group that uses modular arithmetic as its operation.

Definition 3.7. *The finite cyclic group $Z_n = \{0, 1, \ldots, n-1\}$ is an abelian group under addition modulo n.*

For the group Z_n, the identity element is $e = 0$ because $x + 0 = 0 + x = x$ for any $x \in Z_n$. The inverse of x is $-x = n - x$ and the order of x, $ord(x)$, is the smallest integer m such that $mx = 0$.

Now that we have covered all the necessary background terminology, we will introduce Cayley graphs and Cayley maps. Cayley graphs are a way to draw pictures of groups.

Definition 3.8. *Suppose H is a group with n elements and X is a subset of $H - \{e\}$ that is closed with respect to inverses. The Cayley graph $C_G(H, X)$ is a graph on n vertices, labeled by the n elements of H. The edges are determined by X: vertices g and h are adjacent if and only if there exists some $x \in X$ such that $g = h * x$ (where $*$ is the group operation of H).*

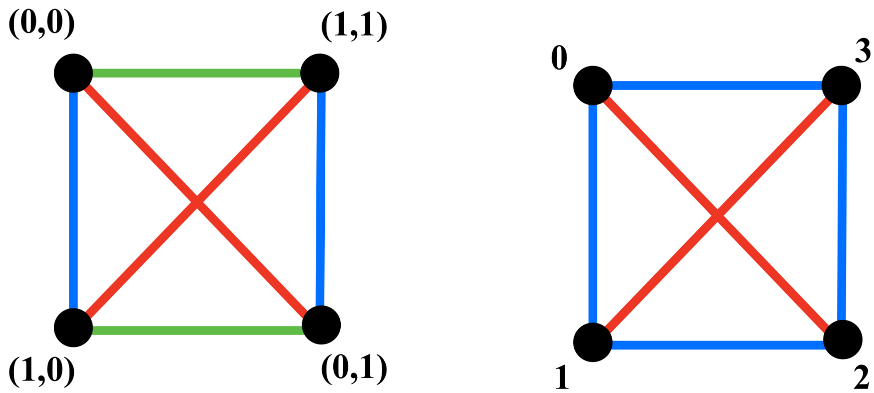
For example, in Figure 2a, the complete graph K_4 is represented by the Cayley graph $C_G(Z_2 \times Z_2, \{(0,1), (1,0), (1,1)\})$, where $Z_2 \times Z_2 = \{(0,0), (0,1), (1,0), (1,1)\}$ is a group under the operation component-wise addition modulo 2. The four vertices are labeled by the four elements of $Z_2 \times Z_2$ and the edges are determined by $\{(0,1), (1,0), (1,1)\}$: the $(0,1)$-edges are red, the $(1,0)$-edges are blue, and the $(1,1)$-edges are green.

We can also represent K_4 using the Cayley graph $C_G(Z_4, \{1, 2, 3\})$. In Figure 2b, the four vertices are labeled by the four elements of Z_4 and the edges are determined by $\{1, 2, 3\}$. The blue edges are 1-edges in one direction and 3-edges in the other direction, and red edges are 2-edges in both directions.

Cayley maps embed Cayley graphs onto orientable surfaces without edge crossings.

Definition 3.9. *Cayley map $C_M(H, \rho)$ embeds Cayley graph $C_G(H, X)$ onto a surface, where X is a closed subset of H and $\rho = (x_1, x_2, \ldots, x_k)$ is a cyclic permutation of X that gives the counter-clockwise rotation of edges around each vertex.*

Although no Cayley map embeds K_4 on a sphere using the group Z_4, an optimal embedding is possible using the group $Z_2 \times Z_2$. In Figure 4a, the Cayley map $C_M(Z_2 \times Z_2, ((0,1), (1,0), (1,1)))$



(A) $C_G(Z_2 \times Z_2, \{(0,1),(1,0),(1,1)\})$ (B) $C_G(Z_4, \{1,2,3\})$

FIGURE 2. Cayley graph representations of K_4

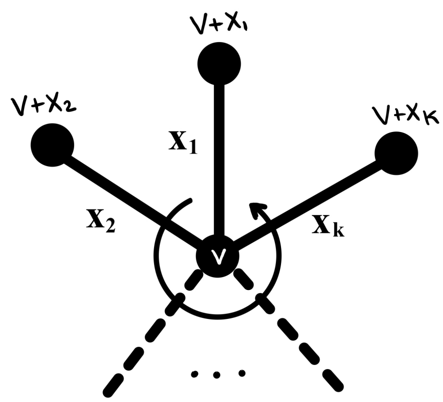
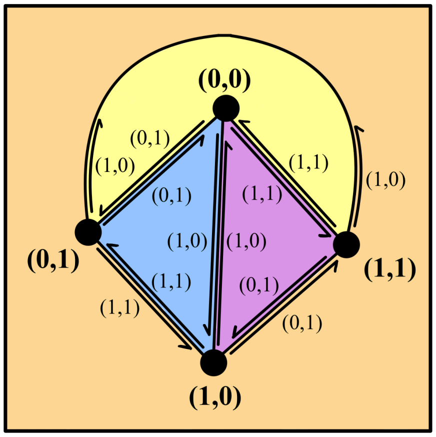


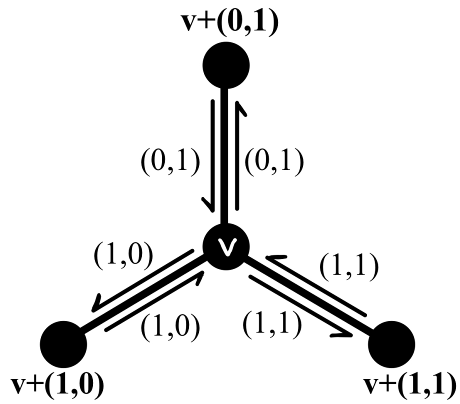
FIGURE 3. The counterclockwise edge rotation around each vertex v in a Cayley map embedding with $\rho = (x_1, x_2, \ldots, x_k)$

optimally embeds $C_G(Z_2 \times Z_2, \{(0,1),(1,0),(1,1)\})$ onto a sphere. Figure 4b shows the counterclockwise rotation $\rho = ((0,1),(1,0),(1,1))$ of each vertex. This embedding has Euler characteristic $\chi = 4 - 6 + 4 = 2$, so K_4 is embedded on a surface with genus $g = 0 = \gamma(K_4)$, giving us an optimal embedding on a sphere using a Cayley map.

Faces generated by Cayley maps are described by disjoint cyclic permutations of their edge types through permutation λ. For example, if (a, b, c) is a cycle in λ, then in a clockwise boundary walk of a face generated by the embedding, any edge of type a is followed by an edge of type b, and likewise any edge of type b is followed by an edge of type c, and any edge of type c is followed by an edge of type a. For example, the Cayley map $C_M(Z_2 \times Z_2, ((0,1),(1,0),(1,1)))$ from Figure 4 has $\lambda = ((0,1),(1,0),(1,1))$ which produces four 3-gons. The relationship between λ and ρ is defined below.



(A) Embedding has four faces



(B) Rotation of each vertex

FIGURE 4. $C_M(Z_2 \times Z_2, ((0,1),(1,0),(1,1)))$ optimally embeds K_4 on a sphere

Definition 3.10. *Suppose H is a group and X is a subset of H that is closed with respect to inverses. Then $\lambda(x) = \rho(x^{-1})$ (and therefore $\rho(x) = \lambda(x^{-1})$).*

If $\rho = (\dots, x, y, \dots)$, then $\rho(x) = y$ and the next edge leaving a vertex counterclockwise from the x-edge is the y-edge. Likewise, if $(\dots, x, y, \dots)$ is a disjoint cycle in λ, then $\lambda(x) = y$ and an edge of type y follows an edge of type x in a clockwise walk along the boundary edges of a face.

If π is a cyclic permutation, the size $|\pi|$ is the number of elements that are not fixed by π. We will call a cyclic permutation π a k-cycle when $|\pi| = k$. If H is a group and $\pi = (x_1, x_2, \dots, x_n)$ is a cyclic permutation of elements of H, the multiplicity of π, denoted $mult(\pi)$, is the order of $x_1 \cdot x_2 \cdots x_n$ in H.

In this paper, we will be primarily focusing on Cayley maps with a prime number p of vertices. Every group with prime number p of elements is isomorphic to the finite cyclic group Z_p. Hence, for complete graph K_p, we have Cayley graph $C_G(Z_p, X)$ embedded by a Cayley map of the form $C_M(Z_p, \rho)$, where ρ is a cyclic permutation of $X = Z_p - \{0\}$. Thus, when looking at any Cayley Map $C_M(Z_n, \rho)$, all arithmetic will be assumed to be addition modulo n. The order of a group element $x \in Z_p$, denoted $ord(x)$, is the smallest integer n such that $nx = 0$. If $\lambda = (x_1, x_2, \dots, x_n)$ is a cyclic permutation of group elements, the multiplicity of λ, $mult(\lambda)$, is the order of the sum $x_1 + x_2 + \cdots + x_n$ in Z_p.

We will write $\lambda = \lambda_1 \cdots \lambda_m$ as a product of m disjoint cyclic permutations. If λ_i is a cyclic permutation of group elements, $Face(\lambda_i) = |\lambda_i| \cdot mult(\lambda_i)$. We will say λ_i generates k-gons when

$Face(\lambda_i) = k$. We determine how many k-gons λ_i generates using $\frac{p|\lambda_i|}{Face(\lambda_i)}$. If $\lambda_i = (x_1, x_2, \ldots, x_n)$ is a cyclic permutation of group elements, $Sum(\lambda_i) = x_1 + x_2 + \cdots + x_n$.

4. Small Complete Graph Embeddings

It is possible to optimally embed several small complete graphs using Cayley maps. We will describe optimal embeddings for K_5, K_6, and K_7 on the torus. In order to better grasp the behavior of these embeddings, we will use the flat torus shown in Figure 5 to visualize these genus 1 embeddings. Surfaces with higher genera can be represented using flat polygons as well [5], but we will just discuss them combinatorially.

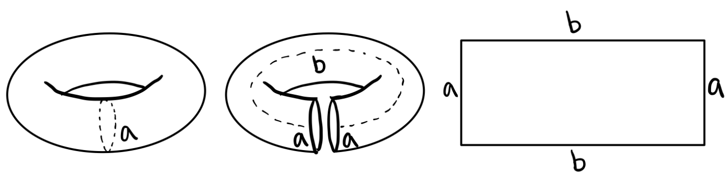
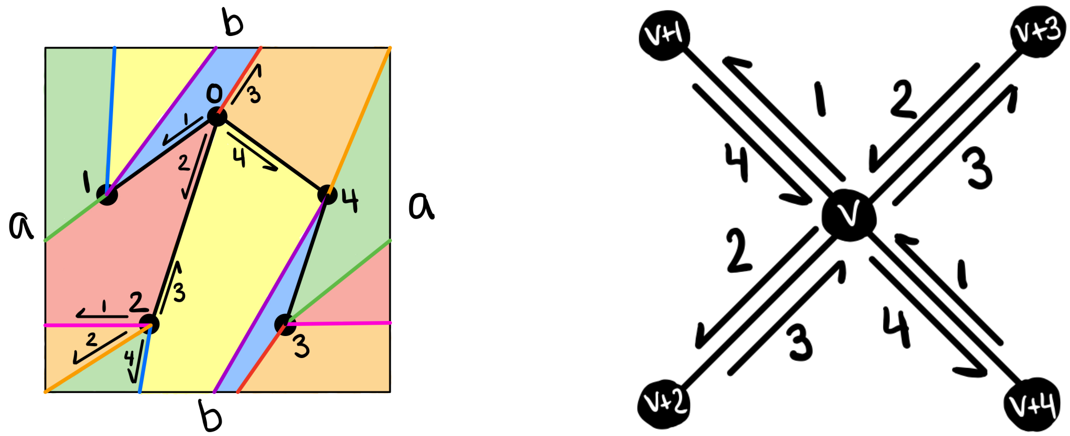


Figure 5. Unrolling the torus into a flat rectangle representation. Edges that go off one side reenter on the corresponding side. The four corners all join at one single point.

We are able to embed K_5 optimally on a torus using the Cayley map $C_M(Z_5, (1, 2, 4, 3))$ which has $\lambda = (1, 3, 4, 2)$. The rotation around each vertex is shown in Figure 6b. This embedding produces five 4-gons, seen in Figure 6a, and has Euler characteristic $\chi = 5 - 10 + 5 = 0$, so the surface has genus $g = 1 = \gamma(K_5)$. Therefore, K_5 is embedded on a torus, which is an optimal Cayley map embedding according to Theorem 3.3.

It is also possible to optimally embed K_6 on a torus using a Cayley map. The Cayley map $C_M(Z_6, (1, 5, 3, 2, 4))$ has $\lambda = (1, 3, 2)(4)(5)$, producing eight 3-gons and one 6-gon for nine faces total, represented in Figure 7a. The vertex rotation is shown in Figure 7b. The resulting Euler characteristic is $\chi = 0$, so this Cayley map optimally embeds K_6 on a torus with genus $\gamma(K_6) = g = 1$.

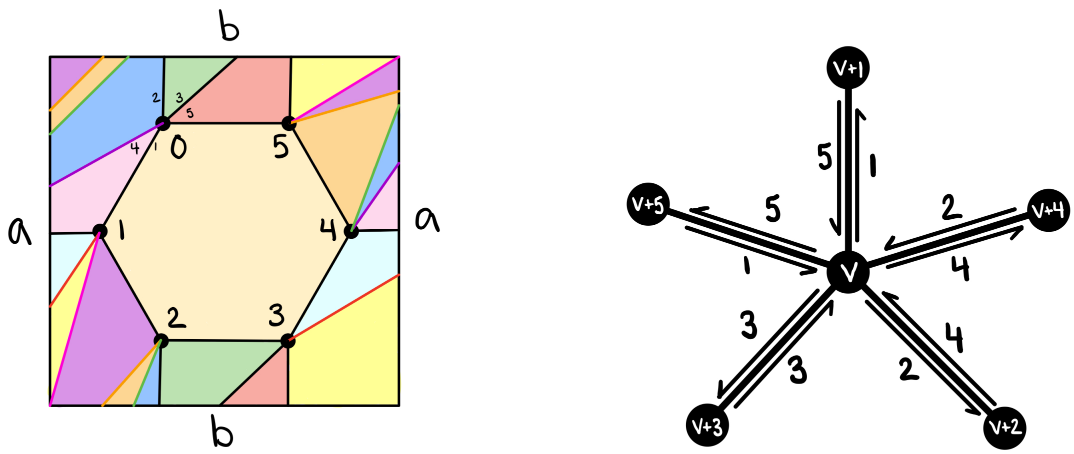
There are two Cayley maps that optimally embed K_7 on a torus. Both generate fourteen 3-gons. One of these Cayley maps is $C_M(Z_7, (1, 3, 2, 6, 4, 5))$ with $\lambda = (3, 5, 6)(1, 4, 2)$. The rotation around each vertex is shown in Figure 8. The flat polygon representation of this embedding of K_7 could be constructed from the given information in the same manner as in the previous examples. The Euler characteristic $\chi = 7 - 21 + 14 = 0$, so the surface has genus $g = 1$. Therefore, K_7 is embedded on a torus, optimal by Theorem 3.3 since $\gamma(K_7) = 1$.



(A) Embedding has five faces

(B) Rotation of each vertex

FIGURE 6. $C_M(Z_5, (1, 2, 4, 3))$ optimally embeds K_5 on a torus



(A) Embedding has nine faces

(B) Rotation of each vertex

FIGURE 7. $C_M(Z_6, (1, 5, 3, 2, 4))$ optimally embeds K_6 on a torus

The complete graphs on five, six, and seven vertices were all optimally embedded using Cayley maps. Recall from the previous section that the complete graph on four vertices was optimally embedded on the sphere using a Cayley map with the group $Z_2 \times Z_2$ instead of Z_4. From now on, we will only need to consider the group Z_n since we will mostly be exploring complete graphs K_n with n prime. In the next section, we will prove several lemmas and theorems that will help us characterize Cayley map embeddings for larger complete graphs, particularly those with prime numbers of vertices.

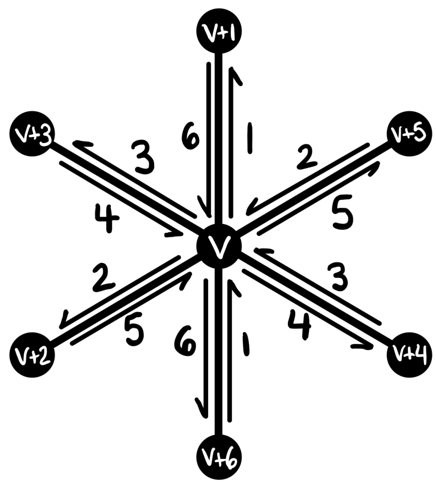


FIGURE 8. Vertex rotation for $C_M(Z_7, (1,3,2,6,4,5))$

5. Initial Lemmas

We now prove several useful lemmas that we will refer to in later sections, primarily Sections 6, 7, and 8, to help us find best possible Cayley map embeddings of some complete graphs. The Inverse Theorem by Spies [10], restated below, will be useful in several of these proofs.

Theorem 5.1. *Suppose H is a group. For a Cayley Map $C_M(H, \rho)$, if a cycle λ_i in λ is closed with respect to inverses, then for all $x \in \rho$, $x \in \lambda_i$.*

This theorem tells us that we can only have a cycle in λ that is closed with respect to inverses if it contains all elements of ρ. The next lemma, which results almost directly from Lagrange's Theorem, will restrict the possible multiplicities of cycles in λ for complete graphs with prime numbers of vertices.

Lemma 5.2. *For a Cayley map $C_M(Z_p, \rho)$ with prime p, any cyclic factor of λ has either multiplicity 1 or multiplicity p.*

Proof. Let λ_i be a cyclic factor of λ for a Cayley map $C_M(Z_p, \rho)$ with prime p. By group closure, $Sum(\lambda_i) = y$ for some $y \in Z_p$. Thus, by definition of multiplicity, $mult(\lambda_i) = ord(y)$ which must divide $|Z_p| = p$ by Lagrange's Theorem. Since p is prime, $mult(\lambda_i) = 1$ or $mult(\lambda_i) = p$. □

Now we will prove several lemmas restricting the elements of cycles in λ. First we show that a 3-cycle of multiplicity 1 cannot contain inverse elements.

Lemma 5.3. *For any Cayley map $C_M(Z_n, \rho)$, no 3-cycle in λ that generates 3-gons can contain inverse elements.*

Proof. Suppose λ_1 is a cyclic factor of λ such that $|\lambda_1| = 3$ and $Face(\lambda_1) = 3$. Let x, y, and z be the three elements of λ_1. Therefore, $mult(\lambda_1) = ord(x+y+z) = 1$. By definition of order of group elements, we know $x + y + z = 0$. If y is the inverse of x, meaning $x + y = 0$, then we have $z = 0$, a contradiction by the definition of $C_M(Z_n, \rho)$. Therefore, λ_1 cannot contain inverse elements. □

Now we look at inverse elements in 4-cycles of multiplicity 1. Unless λ is a cyclic permutation such that $|\lambda| = 4$, no 4-cycle of multiplicity 1 in λ can contain any inverse elements.

Lemma 5.4. *Suppose H is an abelian group. For any Cayley map $C_M(H, \rho)$, if λ_1, a factor of λ, is a 4-cycle that generates 4-gons and $\lambda_1 \neq \lambda$, then λ_1 cannot contain inverse elements.*

Proof. We prove the contrapositive. Suppose λ_1, a factor of λ, is a 4-cycle that permutes the elements $\{a, b, c, d\}$, where $a = -b$. If λ_1 generates 4-gons, meaning $Face(\lambda_1) = 4$, then $mult(\lambda_1) = 1$. This implies $a+b+c+d = 0$. Since $a+b = 0$, by substitution we know $c+d = 0$, meaning d is the inverse of c. Therefore, λ_1 is closed with respect to inverses. By Theorem 5.1, $\lambda_1 = \lambda$. Therefore, if λ_1 is a 4-cycle that generates 4-gons and $\lambda_1 \neq \lambda$, then λ_1 cannot contain inverse elements. □

The next few lemmas use the orbit of one cycle of λ to restrict the elements and orbits of additional cycles of λ. We first show that we only have $\lambda_1 = (x_1, x_2, \ldots, x_k)$ and $\lambda_2 = (-x_1, -x_2, \ldots, -x_k)$ for some odd integer k if λ_1 and λ_2 are the only factors of λ.

Lemma 5.5. *Suppose H is an abelian group, $X = \{x_1, x_2, \ldots, x_n\}$ is a closed subset of H, and ρ is a cyclic permutation of X. For the Cayley Map $C_M(H, \rho)$, if there are at least three factors of λ and $\lambda_1 = (x_1, x_2, \ldots, x_k)$ for some odd integer $k < \frac{n}{2}$, then $\lambda_2 \neq (-x_1, -x_2, \ldots, -x_k)$.*

Proof. We prove the contrapositive. Let $\lambda_1 = (x_1, x_2, \ldots, x_k)$ and $\lambda_2 = (-x_1, -x_2, \ldots, -x_k)$ for some odd $k < \frac{n}{2}$. By Definition 3.10, $\rho(x_1) = -x_2$, $\rho(-x_2) = x_3, \ldots, \rho(-x_{k-1}) = x_k$, $\rho(x_k) = -x_1, \rho(-x_1) = x_2, \ldots, \rho(-x_k) = x_1$, resulting in $\rho = (x_1, -x_2, x_3, \ldots, -x_{k-1}, x_k, -x_1, x_2, -x_3, \ldots, x_{k-1}, -x_k)$. Since all elements of ρ are in λ_1 and λ_2, λ has only two factors, a contradiction. Hence, if λ_1 and λ_2 are not the only two factors of λ and $\lambda_1 = (x_1, x_2, \ldots, x_k)$ for some odd integer $k < \frac{n}{2}$, then $\lambda_2 \neq (-x_1, -x_2, \ldots, -x_k)$. □

Corollary 5.6 is an immediate result of Lemma 5.5 when $k = |\lambda_1| = 3$.

Corollary 5.6. *Suppose H is an abelian group, $X = \{x_1, x_2, \ldots, x_n\}$ is a closed subset of H, and ρ is a cyclic permutation of X. For the Cayley Map $C_M(H, \rho)$, if $\lambda \neq \lambda_1\lambda_2$ and $\lambda_1 = (x_1, x_2, x_3)$, then $\lambda_2 \neq (-x_1, -x_2, -x_3)$.*

Now we expand on Corollary 5.6 by showing that if λ has factors λ_1 and λ_2, both 3-cycles of multiplicity 1, and at least one other factor, then λ_1 can only contain one element that is the inverse of an element in λ_2.

Lemma 5.7. *Suppose λ_1 and λ_2 are factors of λ, where $\lambda \neq \lambda_1\lambda_2$, for Cayley map $C_M(Z_p, \rho)$ with prime p, such that $|\lambda_1| = |\lambda_2| = 3$ and $mult(\lambda_1) = mult(\lambda_2) = 1$. If $\lambda_1 = (a, b, c)$ and $-a \in \lambda_2$, then $-b, -c \notin \lambda_2$*

Proof. Let $\lambda_1 = (a, b, c)$ and $-a \in \lambda_2$. Suppose $-b, x \in \lambda_2$. Since $mult(\lambda_2) = 1$, $-a-b+x = 0$. Also, since $mult(\lambda_1) = 1$, we know $a+b+c = 0$. By the definition of inverse elements, $a+b+c-a-b-c = 0$, implying $-a-b-c = 0$. It follows that $x = -c$ such that $-a, -b, -c \in \lambda_2$. By Corollary 5.6, $\lambda_2 \neq (-a, -b, -c)$. Additionally, if $\lambda_2 = (-a, -c, -b)$, then ρ is not a cyclic permutation, a contradiction. Thus, $-b \notin \lambda_2$. By symmetric argument, $-c \notin \lambda_2$. □

Next we prove several lemmas about the faces generated by λ for a Cayley map $C_M(Z_p, \rho)$ with prime p. Recall that a cycle λ_i in λ generates $\frac{p|\lambda_i|}{k}$ k-gons, where $k = Face(\lambda_i) = |\lambda_i| mult(\lambda_i)$. First we show that any 1-cycle in λ must have multiplicity p, and therefore generates only one face.

Lemma 5.8. *For a Cayley map $C_M(Z_p, \rho)$ with prime p, any 1-cycle in λ has multiplicity p and generates one face, a p-gon.*

Proof. Since p is prime, $ord(x) = p$ for any $x \in \lambda$. Let $\lambda_1 = (x_1)$ be a 1-cycle in λ. Therefore, $mult(\lambda_1) = ord(x_1) = p$ and $Face(\lambda_1) = mult(\lambda_1) = p$. Thus, a λ_1 produces one face, a p-gon. □

Similarly, any 2-cycle in λ generates only one face because it also must have multiplicity p.

Lemma 5.9. *For a Cayley map $C_M(Z_p, \rho)$ with prime p, any 2-cycle in λ has multiplicity p and generates one face, a $2p$-gon.*

Proof. Let $C_M(Z_p, \rho)$ be a Cayley map for K_p with prime p. If $p = 3$, the only possible Cayley map for K_p is $C_M(Z_3, (1, 2))$, which has $\lambda = (1)(2)$. Thus, a Cayley map $C_M(Z_p, \rho)$ with a 2-cycle must have prime $p \geq 5$. Let $\lambda_1 = (x_1, x_2)$ be a 2-cycle in λ. By Lemma 5.2, $mult(\lambda_1) = 1$ or

$mult(\lambda_1) = p$ since p is prime. By Theorem 5.1, $mult(\lambda_1) = ord(x_1 + x_2) \neq 1$, so $mult(\lambda_1) = p$. Hence, $Face(\lambda_1) = 2mult(\lambda_1) = 2p$ and therefore λ_1 generates one face, a $2p$-gon. □

Like 1-cycles and 2-cycles, any cycle in λ with three or more elements generates only one face when it has multiplicity p. However, a cycle of at least size three can also have multiplicity 1 and generate p faces.

Lemma 5.10. *Suppose a Cayley map $C_M(Z_p, \rho)$ with prime p has a factor λ_1 of λ. If $|\lambda_1| \geq 3$, then λ_1 either generates one face or p faces.*

Proof. Let $|\lambda_1| \geq 3$ for $C_M(Z_p, \rho)$ with prime p. By Lemma 5.2, $mult(\lambda_1) = 1$ or $mult(\lambda_1) = p$. If $mult(\lambda_1) = 1$, then $Face(\lambda_1) = |\lambda_1|$, so λ_1 generates $\frac{p|\lambda_1|}{|\lambda_1|} = p$ faces. If $mult(\lambda_1) = p$, then $Face(\lambda_1) = p|\lambda_1|$, so λ_1 generates $\frac{p|\lambda_1|}{p|\lambda_1|} = 1$ face. Therefore, λ_1 generates either one face or p faces. □

By Lemma 5.2, a Cayley map $C_M(Z_p, \rho)$ with prime p can only have cycles of multiplicity 1 and multiplicity p in λ. For a Cayley map $C_M(Z_p, \rho)$ with prime p, we will call λ an (m, n)-permutation when λ has m multiplicity 1 cycles and n multiplicity p cycles.

Theorem 5.11. *If $C_M(Z_p, \rho)$ with prime p is a Cayley map such that λ is an (m, n)-permutation, meaning λ has m cycles of multiplicity 1 and n cycles of multiplicity p, then λ generates $mp + n$ faces.*

Proof. Let $C_M(Z_p, \rho)$ with prime p have an (m, n)-permutation λ. By Lemmas 5.8 and 5.9, any 1-cycle or 2-cycle in λ has multiplicity p and generates one face. By Lemma 5.10, a cycle of at least size 3 has either multiplicity p and generates one face or has multiplicity 1 and generates p faces. Thus, the m multiplicity 1 cycles generate p faces each and the n multiplicity p cycles generate one face each, for a total of $mp + n$ faces. □

An (m, n)-permutation λ for $C_M(Z_p, \rho)$ with prime p generates $mp + n$ faces by Theorem 5.11. Since $|\rho| = p - 1$, it is clear that $0 \leq n < p$ for any (m, n). Additionally, $0 \leq m \leq \lfloor \frac{p-1}{3} \rfloor$ since 3-cycles are the smallest cycle that can have multiplicity 1 by Lemmas 5.8, 5.9, and 5.10. We will say $(a, b) > (c, d)$ when $ap + b > cp + d$, meaning an (a, b)-permutation generates more faces than a (c, d)-permutation.

Theorem 5.12. *For a Cayley map $C_M(Z_p, \rho)$ with prime p, λ can be an (a, b)-permutation or a (c, d)-permutation. If $a > c$ or if $a = c$ and $b > d$, then $(a, b) > (c, d)$.*

Proof. Suppose $a > c$, meaning $a \geq c + 1$ since a and c are integers. Since $p > 0$, multiplying by p gives $ap \geq cp + p$. Clearly $b \geq 0$ so $ap + b \geq ap$. By transitivity, $ap + b \geq cp + p$. Additionally, $cp + p > cp + d$ since $p > p - 1 = |\rho| \geq d$ so $p > d$. By transitivity, $ap + b > cp + d$. Hence, $(a, b) > (c, d)$ when $a > c$.

Now suppose $a = c$ and $b > d$. It immediately follows that $ap = cp$ and thus $ap + b > cp + d$. Therefore, $(a, b) > (c, d)$ when $a = c$ and $b > d$. $\square$

Corollary 5.13 follows from Theorem 5.12 because $(a, b) > (m, n)$ means an (a, b)-permutation generates more faces than an (m, n)-permutation. A Cayley map with λ that generates the maximum possible number of faces is the best possible embedding for that graph.

Corollary 5.13. *Let $C_M(Z_p, \rho)$ with prime p have λ that is an (a, b)-permutation. If $(a, b) \geq (m, n)$ for any other (m, n)-permutation, then $C_M(Z_p, \rho)$ is a best possible Cayley map embedding of K_p.*

By Corollary 5.13, it is clear that a $(\lfloor \frac{p-1}{3} \rfloor, p - 1 - 3\lfloor \frac{p-1}{3} \rfloor)$-permutation is better or as good as any other (m, n)-permutation.

Theorem 5.14. *For a Cayley map $C_M(Z_p, \rho)$ with prime p, if λ is a $(\lfloor \frac{p-1}{3} \rfloor, p - 1 - 3\lfloor \frac{p-1}{3} \rfloor)$-permutation, then $C_M(Z_p, \rho)$ is a best possible Cayley map embedding of K_p.*

Proof. Let $C_M(Z_p, \rho)$ be a Cayley map for for the complete graph K_p with prime p. By definition, ρ is a cyclic permutation of $Z_p - \{0\}$, so we have $|Z_p - \{0\}| = p - 1$ elements in ρ and the disjoint cycles of λ. By Lemmas 5.8 and 5.9, any 1-cycle or 2-cycle in λ has multiplicity p and by Lemma 5.10, a 3-cycle or larger can have multiplicity 1 or multiplicity p. Therefore, the maximum number of multiplicity 1 cycles is $\lfloor \frac{p-1}{3} \rfloor$. Then $p - 1 - 3\lfloor \frac{p-1}{3} \rfloor$ elements are not in multiplicity 1 cycles, so to maximize the number of multiplicity p cycles, each element is in a 1-cycle, resulting in $p - 1 - 3\lfloor \frac{p-1}{3} \rfloor$ multiplicity p cycles. This results in a $(\lfloor \frac{p-1}{3} \rfloor, p - 1 - 3\lfloor \frac{p-1}{3} \rfloor)$-permutation, where $\lfloor \frac{p-1}{3} \rfloor \geq m$ for all (m, n)-permutations, and $p - 1 - 3\lfloor \frac{p-1}{3} \rfloor \geq n$ for all $(\lfloor \frac{p-1}{3} \rfloor, n)$-permutations. Therefore, by Corollary 5.13, a $(\lfloor \frac{p-1}{3} \rfloor, p - 1 - 3\lfloor \frac{p-1}{3} \rfloor)$-permutation is a best possible Cayley map embedding of K_p. $\square$

Now we prove several lemmas about the size and multiplicity of cycles in λ. First we show that if there are m factors of λ and $m-1$ of them are known to have multiplicity 1, then the final factor must also have multiplicity 1.

Lemma 5.15. *For a Cayley map $C_M(Z_n, \rho)$ with odd n, if λ has m cyclic factors and $mult(\lambda_i) = 1$ for all $i \in \{1, \ldots, m-1\}$, then $mult(\lambda_m) = 1$.*

Proof. Let λ have $m \geq 2$ factors and $mult(\lambda_i) = 1$ for all $i \in \{1, \ldots, m-1\}$. Therefore, $Sum(\lambda_i) = 0$ for all i so $\Sigma_{i=1}^{m-1} Sum(\lambda_i) = 0$. Since n is odd, $\Sigma_{k=1}^{n-1} k = 0$. Additionally, $\Sigma_{k=1}^{n-1} k = \Sigma_{i=1}^{m} Sum(\lambda_i) = \Sigma_{i=1}^{m-1} Sum(\lambda_i) + Sum(\lambda_m) = 0$. By substitution $Sum(\lambda_m) = 0$. This implies $mult(\lambda_m) = 1$. □

The corollary below follows directly from Lemma 5.15 when n is prime.

Corollary 5.16. *A Cayley map $C_M(Z_p, \rho)$ with prime p cannot have λ that is a $(k, 1)$-permutation for $k \geq 1$.*

If we know a Cayley map $C_M(Z_p, \rho)$ generates n-gons, then we can determine the size and multiplicity of certain factors of λ depending on whether $n < p$ or $n \geq p$.

Lemma 5.17. *If a Cayley map $C_M(Z_m, \rho)$ has λ that generates n-gons, where $n < m$ and $gcd(m, n) = 1$, then λ has at least one factor, λ_1, such that $|\lambda_1| = n$ and $mult(\lambda_1) = 1$.*

Proof. For λ_1 to generate n-gons, $Face(\lambda_1) = |\lambda_1| \cdot mult(\lambda_1) = n$. Hence, $mult(\lambda_1)$ divides n. Also, by Lagrange's Theorem, $mult(\lambda_1)$ must divide m, the size of the group, so $mult(\lambda_1)$ is a common divisor of m and n. Since $gcd(m, n) = 1$, $mult(\lambda_1) = 1$. This means $|\lambda_1| = n$. Therefore, the only way to have $Face(\lambda_1) = n$ is when $|\lambda_1| = n$ and $mult(\lambda_1) = 1$. □

Corollary 5.18 follows from Lemma 5.17 when the number of n-gons is known, because each λ_i in λ for $C_M(Z_m, \rho)$ generates $\frac{m|\lambda_i|}{n} = \frac{mn}{n} = m$ n-gons. When there are r total n-gons and each λ_i generates m of them, there must be $\frac{r}{m}$ factors λ_i.

Corollary 5.18. *If a Cayley map $C_M(Z_m, \rho)$ has λ that generates r n-gons, where $n < m$ and $gcd(m, n) = 1$, then λ has factors λ_i such that $|\lambda_i| = n$ and $mult(\lambda_i) = 1$ for all $i \in \{1, \ldots, \frac{r}{m}\}$.*

Similarly, Lemma 5.19 tells us about certain factors of λ based on how many n-gons are generated but for $C_M(Z_p, \rho)$ with prime p and $n \geq p$.

Lemma 5.19. *If a Cayley map $C_M(Z_p, \rho)$ with prime p has λ that generates m n-gons, where $n \geq p$, then λ has factors λ_i such that $|\lambda_i| = \frac{n}{p}$ and $mult(\lambda_i) = p$ for all $i \in \{1, \ldots, m\}$.*

Proof. Suppose λ generates m n-gons for some $n \geq p$. Let λ_i be a factor of λ that generates n-gons. Thus, $Face(\lambda_i) = |\lambda_i| \cdot mult(\lambda_i) = n \geq p$ so $|\lambda_i| \cdot mult(\lambda_i) > p - 1$. Since the cycles of λ have $p-1$ distinct elements, $p - 1 \geq |\lambda_i| \geq 1$. By transitivity, $|\lambda_i| \cdot mult(\lambda_i) > |\lambda_i|$, so $mult(\lambda_i) > 1$ and therefore $mult(\lambda_i) = p$ by Lemma 5.2. Thus, $Face(\lambda_i) = p|\lambda_i| = n$ so $|\lambda_i| = \frac{n}{p}$. Also, λ_i generates $\frac{p|\lambda_i|}{Face(\lambda_i)} = \frac{n}{n} = 1$ n-gon, so there must be m such factors λ_i of λ to generate m n-gons. Hence, λ has m factors λ_i such that $|\lambda_i| = \frac{n}{p}$ and $mult(\lambda_i) = p$ for all $i \in \{1, \ldots, m\}$. □

The following lemma uses Definition 3.10, which states $\lambda(x) = \rho(x^{-1})$, to determine the size of cyclic permutation ρ.

Lemma 5.20. *Suppose H is a group, $X = \{x_1, x_2, \ldots, x_n\}$ is a closed subset of H, and ρ is a cyclic permutation of X. For the Cayley Map $C_M(H, \rho)$, if $\lambda(x_1^{-1}) = x_2$, $\lambda(x_2^{-1}) = x_3$, $\lambda(x_3^{-1}) = x_4$, $\ldots$, $\lambda(x_k^{-1}) = x_1$, then ρ is a k-cycle.*

Proof. Let $C_M(H, \rho)$ be a Cayley map with λ such that $\lambda(x_1^{-1}) = x_2$, $\lambda(x_2^{-1}) = x_3$, $\lambda(x_3^{-1}) = x_4$, $\ldots$, $\lambda(x_k^{-1}) = x_1$. By Definition 3.10, $\rho(x_1) = \lambda(x_1^{-1}) = x_2$, $\rho(x_2) = \lambda(x_2^{-1}) = x_3$, $\rho(x_3) = \lambda(x_3^{-1}) = x_4$, $\ldots$, $\rho(x_k) = \lambda(x_k^{-1}) = x_1$. This results in $\rho = (x_1, x_2, x_3, \ldots, x_k)$, a k-cycle. □

Corollary 5.21 follows directly from Lemma 5.20 when $k = 2$.

Corollary 5.21. *Suppose $C_M(H, \rho)$ is a Cayley Map, where H is a group and ρ is a cyclic permutation of elements of H. If $a, b \in \rho$, such that $\lambda(a) = b^{-1}$ and $\lambda(b) = a^{-1}$, then ρ is a 2-cycle.*

In general, a Cayley map $C_M(Z_p, \rho)$ with prime p can achieve the optimal embedding of K_p if it generates only 3-gons.

Theorem 5.22. *If $C_M(Z_p, \rho)$ with prime p is a Cayley map such that λ generates only 3-gons, then $C_M(Z_p, \rho)$ is the optimal Cayley map embedding of K_p.*

Proof. Let $C_M(Z_p, \rho)$ be a Cayley map for for the complete graph K_p with prime p. If $p = 3$ then the only Cayley map embedding of K_p is $C_M(Z_3, (1, 2))$ with $\lambda = (1)(2)$. Thus, λ generates two 3-gons, which results in the optimal genus $g = 0$. Now suppose $p > 3$. By Lemma 5.17, $|\lambda_i| = 3$ and $mult(\lambda_i) = 1$ for all $i \in \{1, 2, \ldots, \frac{p-1}{3}\}$. Hence, λ generates $\frac{p(p-1)}{3}$ faces. Using the

Euler characteristic, we get $\chi = p - \frac{p(p-1)}{2} + \frac{p(p-1)}{3} = \frac{-p^2+7p}{6}$. Then using the genus formula, $g = \frac{p^2-7p+12}{12} = \frac{(p-3)(p-4)}{12} = \gamma(K_p)$, the optimal genus by Definition 3.3. Therefore, a Cayley map $C_M(Z_p, \rho)$ with λ that generates only 3-gons must be the optimal embedding. □

These lemmas, theorems, and corollaries will be used in the next few sections to prove that certain Cayley map embeddings are best possible despite being nonoptimal.

6. Best Possible Cayley Map for K_{11}

By Theorem 3.3, the optimal genus of an embedding of K_{11} is $\gamma(K_{11}) = 5$. Using the Euler characteristic and genus formula for orientable surfaces, we see that a genus of 5 would require the embedding to generate at least 36 faces. However, since each cycle of λ either produces one face or eleven faces, and eleven faces is only possible from a cycle with at least three elements, it is clear that a Cayley map for K_{11} can only produce a maximum of thirty-four faces. Thus, the optimal embedding of K_{11} on a 5-holed torus is impossible with a Cayley map. In fact, we will prove that using Cayley maps, the simplest surface we are able to embed K_{11} on is 10-holed torus with $g = 10$. We achieve a best possible embedding using the Cayley map $C_M(Z_{11}, (1, 7, 4, 5, 6, 10, 2, 9, 8, 3))$, but before we prove this in Theorem 6.2, we must show that no Cayley map embedding of K_{11} has only 3-gons and 4-gons as faces.

Lemma 6.1. *No Cayley map embedding of K_{11} produces only 3-gons and 4-gons.*

Proof. Since the only group with eleven elements is Z_{11}, let $C_M(Z_{11}, \rho)$ be a Cayley map embedding of K_{11}, where ρ has 10 elements, composed of five nonzero elements and their inverses: $a, b, c, d, e, -a, -b, -c, -d, -e$.

Since $3 < p$ and $4 < p$, by Lemma 5.17, to get 3-gons and 4-gons we need at least one 3-cycle, called λ_1 and at least one 4-cycle, called λ_3, such that $mult(\lambda_1) = mult(\lambda_3) = 1$. Note that $|\lambda_1| = 3$ and $|\lambda_3| = 4$, such that $|\lambda_1| + |\lambda_3| = 7$. Since the cycles of λ have ten distinct elements, and $10 - |\lambda_1| - |\lambda_3| = 3$ unused elements, there must be another factor of λ, called λ_2, such that $|\lambda_2| = 3$. Thus, λ_2 must be a 3-cycle that generates 3-gons and $mult(\lambda_2) = 1$. Hence, there are three factors of λ, called λ_1, λ_2, and λ_3, such that λ_1 and λ_2 are 3-cycles, λ_3 is a 4-cycle, and $mult(\lambda_1) = mult(\lambda_2) = mult(\lambda_3) = 1$. By Lemmas 5.3 and 5.4, we know there can be no inverses within λ_1, λ_2, or λ_3.

Let $a, b, c, d \in \lambda_3$. Since λ_1 and λ_2 also cannot have inverses, without loss of generality we can assume $\lambda_1 = (-a, -c, -e)$ and $\lambda_2 = (-b, -d, e)$. Since $\lambda(-a) = -c$, by Corollary 5.21, $\lambda(c) \neq a$. Therefore either $\lambda(c) = b$ or $\lambda(c) = d$.

First suppose $\lambda(c) = d$. Clearly $\lambda(d) \neq c$ and $\lambda(d) \neq d$ since $|\lambda_3| = 4$. By Corollary 5.21, $\lambda(d) \neq b$ since $\lambda(-b) = -d$. Thus, $\lambda(d) = a$ so $\lambda_3 = (c, d, a, b)$. However, then $\rho = (a, -c, d, e, -a, b, -d)(c, -e, -b)$ by Definition 3.10, a contradiction since ρ is cyclic. Hence, $\lambda(c) \neq d$.

Now suppose $\lambda(c) = b$. By Definition 3.10, $\rho = (\dots, d, e, -a, \dots)$. Thus, $\lambda(a) \neq d$, resulting in $\lambda_3 = (c, b, d, a)$. By Definition 3.10, $\rho = (a, -c, b, -d)(c, -e, -b, d, e, -a)$, which contradicts ρ being a cyclic permutation. Hence, $\lambda(c) \neq b$.

Therefore, there is no possible rotation for λ_3, so there is no rotation ρ for K_{11} producing only 3-gons and 4-gons. □

Now that we have shown that no Cayley map embedding of K_{11} produces only 3-gons and 4-gons, we are ready to prove that the best possible embedding generates 26 faces and has genus 10.

Theorem 6.2. *A best possible Cayley map embedding of K_{11} generates twenty-two 3-gons and four 11-gons and is on a 10-holed torus.*

Proof. The only group to consider for K_{11} is Z_{11}. Therefore, the cycles in λ have 10 distinct elements. By Theorem 5.14, a Cayley map embedding of K_{11} by $C_M(Z_{11}, \rho)$ with a $(3,1)$-permutation λ is best possible, but such a λ contradicts Corollary 5.16 and is not possible. The next best λ is a $(3,0)$-permutation, which is only possible by having two 3-cycles of multiplicity one and one 4-cycle of multiplicity one, which would generate only 3-gons and 4-gons and thus contradict Lemma 6.1. Then by Corollary 5.13, a $(2,4)$-permutation is the next best. This is possible with the Cayley map $C_M(Z_{11}, (1, 7, 4, 5, 6, 10, 2, 9, 8, 3))$ with $\lambda = (1, 2, 8)(5, 10, 7)(3)(4)(6)(9)$ that produces 26 faces: twenty-two 3-gons and four 11-gons. Thus, a best possible Cayley map embedding of K_{11} has λ that is a $(2,4)$-permutation and generates twenty-two 3-gons and four 11-gons, embedding it on a ten-holed torus. □

7. Best Possible Cayley Map for K_{13}

By Theorem 3.3, the optimal genus of an embedding of K_{13} is $\gamma(K_{13}) = 8$. Simple algebra using the Euler characteristic and genus formula for an orientable surface shows that the only way to achieve this genus would be with an embedding that generates at least fifty-one faces. Since each cycle of λ either produces one face or thirteen faces, it is clear that the only such embedding for K_{13} would have λ with four 3-cycles of multiplicity 1 to generate fifty-two faces. However, we will show that this optimal embedding is not possible with a Cayley map, and that the best embedding we can do generates only forty-one faces with a genus of 13.

Theorem 7.1. *No Cayley map embedding of K_{13} produces only 3-gons.*

Proof. Let $C_M(Z_{13}, \rho)$ be a Cayley map for K_{13} that produces only 3-gons. By Corollary 5.18, λ_1, λ_2, λ_3 and λ_4 are factors of λ such that $|\lambda_i| = 3$ and $mult(\lambda_i) = 1$ for $i \in \{1, 2, 3, 4\}$. There are twelve nonzero elements in ρ: $a, b, c, d, e, f, -a, -b, -c, -d, -e, -f$. By Lemma 5.3, there can be no inverse elements within any λ_i. Thus, let $\lambda_1 = (a, b, c)$. By Lemma 5.7, $-a$, $-b$, and $-c$ must be in separate factors of λ. Therefore, suppose $-a \in \lambda_2$, $-b \in \lambda_3$, and $-c \in \lambda_4$. Without loss of generality, let $\lambda_2 = (-a, d, e)$. Then by Lemma 5.7, $-d$ and $-e$ must be in separate λ_i.

Suppose $-e \in \lambda_3$ and $-d \in \lambda_4$. Without loss of generality, $f \in \lambda_3$ and $-f \in \lambda_4$ and so $-b, -e, f \in \lambda_3$ and $-c, -d, -f \in \lambda_4$. By Definition 3.10, $\rho = (\dots, -c, a, d, \dots)$ so $\lambda(-d) \neq -c$ since ρ is not a 3-cycle. Thus, $\lambda_4 = (-d, -f, -c)$. Now $\rho = (\dots, -e, -a, b, \dots)$ by Definition 3.10, so $\lambda(-b) \neq -e$. Thus, $\lambda_3 = (-b, f, -e)$ so $\lambda = (a, b, c)(-a, d, e)(-b, f, -e)(-d, -f, -c)$, but then by Definition 3.10, $\rho = (a, d, -f, -e, -a, b, f, -c)(c, -d, e, -b)$, a contradiction since ρ must be a cyclic permutation. Hence, we cannot have $-e \in \lambda_3$ and $-d \in \lambda_4$.

Now suppose $-d \in \lambda_3$ and $-e \in \lambda_4$. Without loss of generality, $f \in \lambda_3$ and $-f \in \lambda_4$ so $-b, -d, f \in \lambda_3$ and $-c, -e, -f \in \lambda_4$. If $\lambda_3 = (-b, -d, f)$, by Definition 3.10, $\rho = (\dots, -c, a, d, f, \dots)$. Since ρ is not a 4-cycle, $\lambda(-f) \neq -c$, so $\lambda_4 = (-f, -e, -c)$, resulting in $\lambda = (a, b, c)(-a, d, e)$ $(-b, -d, f)(-f, -e, -c)$. Then by Definition 3.10, $\rho = (a, d, f, -e, -a, b, -d, e, -c)(c, -f, -b)$, a contradiction since ρ is a cyclic permutation. If $\lambda_3 = (-d, -b, f)$, by Definition 3.10, $\rho = (\dots, -e, -a, b, f, \dots)$. Then $\lambda(-f) \neq -e$, so $\lambda_4 = (-f, -c, -e)$, resulting in $\lambda = (a, b, c)(-a, d, e)$ $(-d, -b, f)(-f, -c, -e)$. Therefore, by Definition 3.10, $\rho = (a, d, -b, c, -e, -a, b, f, -c)(e, -f, -d)$, another contradiction since ρ is a cyclic permutation. Therefore, we cannot have $-d \in \lambda_3$ and $-e \in \lambda_4$.

Hence, there is no rotation ρ such that $C_M(Z_{13}, \rho)$ produces only 3-gons. $\square$

We just showed that no Cayley map embedding of K_{13} generates only 3-gons, meaning it cannot have all multiplicity one 3-cycles. Now we need to show that it also cannot have thirty-nine 3-gons and three 13-gons.

Theorem 7.2. *No Cayley map embedding of K_{13} produces thirty-nine 3-gons and three 13-gons.*

Proof. Suppose $C_M(Z_{13}, \rho)$ generates thirty-nine 3-gons and three 13-gons. By Corollary 5.18 and Lemma 5.19, λ must have three factors λ_i such that $|\lambda_i| = 3$ and $mult(\lambda_i) = 1$ for $i = 1, 2, 3$ and three factors λ_k such that $|\lambda_k| = \frac{13}{13} = 1$ and $mult(\lambda_k) = 13$ for $k = 4, 5, 6$. There are 12 elements in ρ, which we will call $a, b, c, d, e, f, -a, -b, -c, -d, -e, -f$. By Lemma 5.3, there can be no inverse elements within any λ_i, so let $\lambda_1 = (a, b, c)$.

If $\lambda_4 = (-a)$, $\lambda_5 = (-b)$, and $\lambda_6 = (-c)$, then by definition 3.10, $\rho = (a, -a, b, -b, c, -c)$, a contradiction since ρ is not a 6-cycle. Now suppose $-a \in \lambda_2$, $\lambda_4 = (-b)$, and $\lambda_5 = (-c)$. Without loss of generality, $-a, d, e \in \lambda_2$. By Lemma 5.7, either $\lambda_6 = (-d)$ and $-e \in \lambda_3$ or $\lambda_6 = (-e)$ and $-d \in \lambda_3$. Either way, this means $f, -f \in \lambda_3$, a contradiction by Lemma 5.3.

Therefore, a maximum of one of $-a$, $-b$, and $-c$ can be in a 1-cycle. By Lemma 5.7, no two of $-a$, $-b$, and $-c$ can be in the same 3-cycle. Thus, suppose $-a \in \lambda_2$, $-b \in \lambda_3$, and $\lambda_4 = (-c)$. Without loss of generality, let $\lambda_2 = (-a, d, e)$. This gives us $\lambda = (a, b, c)(-a, d, e)\lambda_3(-c)\lambda_5\lambda_6$ where $-b \in \lambda_3$.

First suppose $-d \in \lambda_3$, which means $\lambda_5 = (-e)$ by Lemma 5.3. Without loss of generality, let $f \in \lambda_3$ meaning $\lambda_6 = (-f)$ by Lemma 5.3. Thus, we have $\lambda = (a, b, c)(-a, d, e)\lambda_3(-c)(-e)(-f)$ where $-b, -d, f \in \lambda_3$. By Definition 3.10, $\rho = (\dots, -d, e, -e, -a, b, \dots)$. Since ρ is not a 5-cycle, $\lambda(-b) \neq -d$, resulting in $\lambda_3 = (-b, f, -d)$. However, then $\rho = (a, d, -b, c, -c)(b, f, -f, -d, e, -e, -a)$, a contradiction since ρ is a cyclic permutation.

Now suppose $-e \in \lambda_3$, which means $\lambda_5 = (-d)$ by Lemma 5.3. Without loss of generality, let $f \in \lambda_3$ meaning $\lambda_6 = (-f)$ by Lemma 5.3. Thus, we have $\lambda = (a, b, c)(-a, d, e)\lambda_3(-c)(-d)(-f)$ where $-b, -e, f \in \lambda_3$. By Definition 3.10, $\rho = (\dots, -e, -a, b, \dots)$, so $\lambda(-b) \neq -e$ since ρ is not a 3-cycle. Thus, $\lambda_3 = (-b, f, -e)$. By Definition 3.10, $\rho = (a, d, -d, e, -b, c, -c)(b, f, -f, -e, -a)$, a contradiction since ρ is a cyclic permutation.

Hence, there is no λ for $C_M(Z_{13}, \rho)$ that generates thirty-nine 3-gons and three 13-gons $\square$

Now we find the genus of a best possible Cayley map embedding of K_{13}.

Theorem 7.3. *A best possible Cayley map embedding of K_{13} generates thirty-nine 3-gons, one 13-gon, and one 26-gon and is on a 13-holed torus.*

Proof. The only group to consider for K_{13} is Z_{13}. By Theorem 5.14, a Cayley map embedding of K_{13} by $C_M(Z_{13}, \rho)$ with a $(4,0)$-permutation λ would be best possible, and in fact optimal by Theorem 5.22 since it must have four 3-cycles of multiplicity 1 and thus generate only 3-gons. However, this is not possible by Theorem 7.1. The next best λ is a $(3,3)$-permutation, but by Theorem 7.2, we cannot have three 3-cycles of multiplicity 1 and three 1-cycles, as this would generate thirty-nine 3-gons and three 13-gons. The next best λ is a $(3,2)$-permutation, which is achieved by the Cayley map $C_M(Z_{13}, (1,3,7,5,4,10,11,2,12,8,6,9))$ with $\lambda = (1,8,4)(3,11,12)(7,9,10)(5,6)(2)$. This is a best possible Cayley map embedding of K_{13}, generating forty-one faces – thirty-nine 3-gons, one 13-gon, and one 26-gon – and with resulting Euler characteristic $\chi = -24$ and genus $g = 13$. Hence, a best possible embedding is on a 13-holed torus. □

8. Best Possible Cayley Map for K_{17}

The optimal genus of an embedding of K_{17} is $\gamma(K_{17}) = 16$ by Theorem 3.3. However, it is impossible to achieve this optimal embedding of K_{17} using a Cayley map. We will show that a best possible Cayley map embedding of K_{17} is on an 18-holed torus rather than the optimal 16-holed torus.

Theorem 8.1. *A best possible Cayley map embedding of K_{17} generates sixty-eight 3-gons and seventeen 4-gons and is on an 18-holed torus.*

Proof. The only group we must consider for K_{17} is Z_{17}. By Theorem 5.14, a Cayley map embedding of K_{17} by $C_M(Z_{17}, \rho)$ with a $(5,1)$-permutation λ would be best possible, but such a λ is not possible by Corollary 5.16. The next best λ is a $(5,0)$-permutation, achieved by $C_M(Z_{17}, (1,7,13,12,15,9,16,4,11,2,14,8,6,10,3,5))$ with $\lambda = (1,4,12)(10,13,11)(14,5,15)(9,6,2)(16,7,3,8)$. This best possible Cayley map embedding of K_{17} generates sixty-eight 3-gons and seventeen 4-gons for a total of eighty-five faces. The resulting Euler characteristic is $\chi = -34$ with genus $g = 18$, so a best possible embedding of K_{17} is on an 18-holed torus. □

9. Optimal Embeddings for Complete Graphs K_{12k+7}

For complete graphs K_n, where $n = 1 \mod 3$, all nonzero elements can be separated into groups of three. If the elements within each of these groups of three add to zero and can be arranged into

3-cycles of λ that define a cyclic ρ, then the graph can be embedded optimally by a Cayley map, per Theorem 5.22. However, as seen in previous sections, not all complete graphs of this form actually achieve an optimal embedding. In this section, we narrow our search for optimal embeddings of complete graphs K_n with $n = 1 \mod 3$ to those where $n = 12m + 7$ for nonnegative integers m. Optimal embeddings were indeed found for complete graphs of the form K_{12m+7} for $0 \leq m \leq 9$.

TABLE 1. Cayley Map Embeddings of Complete Graphs K_{12m+7}

K_{12m+7}	Cayley Map	Faces	Genus	Optimal Genus
K_7	$C_M(Z_7, (1, 5, 4, 6, 2, 3))$ $\lambda = (1, 2, 4)(3, 6, 5)$	14 3-gons	1	1
K_{19}	$C_M(Z_{19}, (1, 6, 15, 7, 17, 16, 5, 18, 2, 9, 13, 14, 11, 4, 10, 12, 8, 3))$ $\lambda = (1, 2, 16)(3, 5, 11)(4, 7, 8)(6, 14, 18)(9, 12, 17)(10, 13, 15)$	114 3-gons	20	20
K_{31}	$C_M(Z_{31}, (1, 13, 27, 24, 8, 25, 20, 22, 10, 26, 6, 14, 18, 19, 9, 29, 28, 4,$ $17, 23, 16, 21, 12, 30, 2, 11, 5, 15, 7, 3))$ $\lambda = (1, 2, 28)(3, 4, 24)(5, 6, 20)(7, 8, 16)(9, 10, 12)(11, 22, 29)$ $(13, 19, 30)(14, 23, 25)(15, 21, 26)(17, 18, 27)$	310 3-gons	63	63
K_{43}	$C_M(Z_{43}, (1, 19, 9, 35, 28, 30, 14, 39, 36, 8, 17, 37, 32, 12, 33, 24, 25,$ $29, 16, 38, 6, 23, 11, 5, 21, 31, 20, 26, 34, 10, 22, 27, 13, 41, 40, 4, 18,$ $42, 2, 15, 7, 3))$ $\lambda = (1, 2, 40)(3, 4, 36)(5, 6, 32)(7, 8, 28)(9, 10, 24)(11, 12, 20)$ $(13, 14, 16)(15, 30, 41)(17, 34, 35)(18, 29, 39)(19, 25, 42)(21, 27, 38)$ $(22, 31, 33)(23, 26, 37)$	602 3-gons	130	130
K_{55}	$C_M(Z_{55}, (1, 19, 9, 23, 11, 5, 29, 39, 24, 50, 6, 27, 13, 43, 32, 41, 28, 34,$ $38, 18, 54, 2, 22, 47, 40, 16, 45, 36, 37, 20, 53, 52, 4, 21, 49, 44, 12, 25,$ $33, 35, 17, 51, 48, 8, 30, 42, 14, 46, 10, 26, 31, 15, 7, 3))$ $\lambda = (1, 2, 52)(3, 4, 48)(5, 6, 44)(7, 8, 40)(9, 10, 36)(11, 12, 32)$ $(13, 14, 28)(15, 16, 24)(17, 18, 20)(19, 37, 54)(21, 38, 51)(22, 35, 53)$ $(23, 41, 46)(25, 42, 43)(26, 39, 45)(27, 34, 49)(29, 31, 50)(30, 33, 47)$	990 3-gons	221	221

K_{67}	$C_M(Z_{67},(1,22,65,64,4,29,39,51,21,66,2,24,11,5,25,63,60,8,$ $26,58,10,38,42,47,32,41,49,37,16,55,18,59,52,20,62,6,33,14,$ $31,44,27,61,56,13,36,50,23,54,43,45,46,30,12,28,57,48,34,$ $40,17,53,19,9,35,15,7,3))$ $\lambda=(1,2,64)(3,4,60)(5,6,56)(7,8,52)(9,10,48)(11,13,43)$ $(12,18,37)(14,19,34)(15,20,32)(16,21,30)(17,23,27)(22,46,66)$ $(24,45,65)(25,47,62)(26,49,59)(28,51,55)(29,42,63)(31,50,53)$ $(33,40,61)(35,41,58)(36,44,54)(38,39,57)$	1474 3-gons	336	336
K_{79}	$C_M(Z_{79},(1,21,77,76,4,31,70,10,43,17,71,64,22,63,24,11,5,33,$ $69,60,28,74,6,35,49,20,78,2,23,61,27,75,72,8,25,67,14,44,50,$ $30,65,53,36,46,51,32,66,55,39,48,52,34,47,19,9,40,16,38,56,$ $58,59,29,73,68,13,45,18,41,57,42,54,62,26,12,37,15,7,3))$ $\lambda=(1,2,76)(3,4,72)(5,6,68)(7,8,64)(9,10,60)(11,13,55)$ $(12,14,53)(15,22,42)(16,24,39)(17,26,36)(18,27,34)(19,28,32)$ $(20,29,30)(21,59,78)(23,58,77)(25,62,71)(31,52,75)(33,51,74)$ $(35,50,73)(37,54,67)(38,57,63)(40,48,70)(41,56,61)(43,46,69)$ $(44,49,65)(45,47,66)$	2054 3-gons	475	475
K_{91}	$C_M(Z_{91},(1,27,13,49,57,24,79,68,51,69,30,86,6,38,77,64,65,$ $46,66,48,20,45,19,9,44,54,70,29,89,88,4,32,85,80,12,36,78,$ $14,52,22,73,25,71,28,87,84,8,42,55,67,33,74,23,11,5,35,82,$ $10,47,56,61,39,53,59,63,43,18,40,17,50,21,75,60,62,41,58,$ $34,83,76,16,37,81,72,26,90,2,31,15,7,3))$ $\lambda=(1,2,88)(3,4,84)(5,6,80)(7,8,76)(9,10,72)(11,12,68)$ $(13,14,64)(15,16,60)(17,23,51)(18,25,48)(19,26,46)(20,28,43)$ $(21,29,41)(22,30,39)(24,33,34)(27,65,90)(31,62,89)(32,63,87)$ $(35,61,86)(36,67,79)(37,70,75)(38,59,85)(40,69,73)(42,57,83)$ $(44,56,82)(45,66,71)(47,54,81)(49,55,78)(50,58,74)(52,53,77)$	2730 3-gons	638	638

K_{103}	$C_M(Z_{103},(1,26,101,100,4,37,82,33,99,96,8,44,17,57,73,54,21,$ $58,23,11,5,39,90,14,55,75,77,78,38,91,80,35,97,92,12,50,74,$ $56,22,94,10,42,85,29,79,36,95,88,16,46,63,25,102,2,28,83,32,$ $93,84,30,87,72,81,34,98,6,41,89,76,59,67,43,61,71,51,64,69,$ $47,18,60,24,53,65,40,86,27,13,52,20,48,62,68,45,66,70,49,$ $19,9,31,15,7,3))$ $\lambda=(1,2,100)(3,4,96)(5,6,92)(7,8,88)(9,10,84)(11,12,80)$ $(13,14,76)(15,16,72)(17,27,59)(18,29,56)(19,30,54)(20,32,51)$ $(21,33,49)(22,34,47)(23,35,45)(24,36,43)(25,38,40)(26,78,102)$ $(28,77,101)(31,81,94)(37,70,99)(39,69,98)(41,68,97)(42,71,93)$ $(44,67,95)(46,73,87)(48,75,83)(50,65,91)(52,64,90)(53,74,79)$ $(55,62,89)(57,63,86)(58,66,82)(60,61,85)$	3502 3-gons	825	825
K_{115}	$C_M(Z_{115},(1,29,113,112,4,30,109,104,12,53,70,75,50,68,77,$ $54,20,63,24,107,100,16,60,73,46,94,35,93,37,101,88,42,102,$ $14,51,18,65,25,58,80,59,22,57,82,64,78,56,21,67,26,111,108,$ $8,32,66,76,52,72,44,99,84,86,87,43,95,34,83,91,39,105,96,$ $79,85,89,41,103,92,38,106,10,49,81,61,23,11,5,45,98,19,9,$ $47,97,33,90,40,110,6,36,17,62,74,48,69,27,13,55,71,28,114,$ $2,31,15,7,3))$ $\lambda=(1,2,112)(3,4,108)(5,6,104)(7,8,100)(9,10,96)(11,12,92)$ $(13,14,88)(15,16,84)(17,19,79)(18,33,64)(20,34,61)(21,35,59)$ $(22,37,56)(23,38,54)(24,39,52)(25,40,50)(26,41,48)(27,42,46)$ $(28,43,44)(29,87,114)(30,89,111)(31,86,113)(32,91,107)$ $(36,85,109)(45,75,110)(47,77,106)(49,76,105)(51,78,101)$ $(53,74,103)(55,73,102)(57,80,93)(58,82,90)(60,71,99)$ $(62,70,98)(63,72,95)(65,68,97)(66,81,83)(67,69,94)$	4370 3-gons	1036	1036

Supposing this pattern of the existence of optimal embeddings continues for all complete graphs of the form K_{12m+7}, we raise the following conjecture.

Conjecture 9.1. *For all $m \geq 0$, there is a cyclic rotation ρ of $\{1,2,3,\ldots,12m+6\}$ such that $C_M(Z_{12m+7},\rho)$ is an optimal embedding of K_{12m+7}.*

See Section 13, Appendix C, for the Python code developed to support this conjecture.

10. General Bounds for Cayley Map Embeddings of Complete Graphs

As we have seen, it is not always possible to embed a complete graph on the optimal surface using a Cayley map. Below, we come up with some bounds to narrow down on how well Cayley map embeddings can do within their constraints. First we find an upper bound on the number of faces a Cayley map embedding of a complete graph with a prime number of vertices can generate.

Theorem 10.1. *For a complete graph K_p with prime p, a Cayley map $C_M(Z_p, \rho)$ generates at most $p-1+(p-3)\lfloor\frac{p-1}{3}\rfloor$ faces.*

Proof. Let $C_M(Z_p, \rho)$ be a Cayley map for for the complete graph K_p with prime p. Let λ be a $(\lfloor\frac{p-1}{3}\rfloor, p-1-3\lfloor\frac{p-1}{3}\rfloor)$-permutation, so by Theorem 5.14, $C_M(Z_p, \rho)$ is a best possible Cayley map embedding of K_p. By Theorem 5.11, since we have $\lfloor\frac{p-1}{3}\rfloor$ multiplicity 1 cycles and $p-1-3\lfloor\frac{p-1}{3}\rfloor$ multiplicity p cycles, λ generates $p\lfloor\frac{p-1}{3}\rfloor + p - 1 - 3\lfloor\frac{p-1}{3}\rfloor = p-1+(p-3)\lfloor\frac{p-1}{3}\rfloor$ faces. Since $C_M(Z_p, \rho)$ is the best possible Cayley map embedding of K_p, no Cayley map embedding of K_p can generate more than $p-1+(p-3)\lfloor\frac{p-1}{3}\rfloor$ faces. □

Now we translate this upper bound on the number of faces into a lower bound for the genus.

Theorem 10.2. *For a complete graph K_p with prime p, an optimal Cayley map $C_M(Z_p, \rho)$ embeds K_p on a surface with genus $g \geq \lceil\frac{p-3}{2}(\frac{p-2}{2} - \lfloor\frac{p-1}{3}\rfloor)\rceil$.*

Proof. Let $C_M(Z_p, \rho)$ be a Cayley map for for the complete graph K_p with prime p. By definition, K_p has p vertices and $\frac{p(p-1)}{2}$ edges. For a Cayley map $C_M(Z_p, \rho)$ that generates F faces, we have Euler characteristic $\chi = p - \frac{p(p-1)}{2} + F$. By Theorem 10.1, $C_M(Z_p, \rho)$ generates a maximum of $p-1+(p-3)\lfloor\frac{p-1}{3}\rfloor$ faces, so $\chi \leq p - \frac{p(p-1)}{2} + p - 1 + (p-3)\lfloor\frac{p-1}{3}\rfloor = 2p-1-\frac{p(p-1)}{2} + (p-3)\lfloor\frac{p-1}{3}\rfloor$. Using the genus formula and transitivity, we have $2p-1-\frac{p(p-1)}{2}+(p-3)\lfloor\frac{p-1}{3}\rfloor \geq 2-2g$ for genus g. Solving for g, we find that $g \geq -p + \frac{3}{2} + \frac{p(p-1)}{4} - \frac{p-3}{2}\lfloor\frac{p-1}{3}\rfloor = \frac{p-3}{2}(\frac{p-2}{2} - \lfloor\frac{p-1}{3}\rfloor)$. Since g must be an integer, $g \geq \lceil\frac{p-3}{2}(\frac{p-2}{2} - \lfloor\frac{p-1}{3}\rfloor)\rceil$. □

For complete graphs K_p where prime $p = 3k+2$ for some integer k, we can improve the bound on the number of faces a Cayley map embedding can generate.

Theorem 10.3. *For a complete graph K_p with prime p, such that $p = 3k+2$ for some integer k, a Cayley map $C_M(Z_p, \rho)$ generates at most $\frac{p(p-2)}{3}$ faces.*

Proof. Let $C_M(Z_p, \rho)$ be a Cayley map for for the complete graph K_p with prime $p = 3k+2$ for some integer k. By definition, ρ is a cyclic permutation of $Z_p - \{0\}$, so we have $|Z_p - \{0\}| = p-1 = 3k+1$ elements in ρ and the cycles of λ. By Theorem 5.14, $C_M(Z_p, \rho)$ is best possible when λ is a $(\lfloor \frac{p-1}{3} \rfloor, p-1-3\lfloor \frac{p-1}{3} \rfloor) = (k,1)$-permutation. However, a $(k,1)$-permutation is not possible by Corollary 5.16. By Corollary 5.13, the next best λ is a $(k,0)$-permutation. Then by Theorem 5.11, $C_M(Z_p, \rho)$ generates $p\lfloor \frac{p-1}{3} \rfloor - p + p = p\lfloor \frac{p-1}{3} \rfloor$ faces. By substitution, $\lfloor \frac{p-1}{3} \rfloor = \lfloor \frac{3k+1}{3} \rfloor = \frac{3k}{3} = \frac{p-2}{3}$, so $p\lfloor \frac{p-1}{3} \rfloor = \frac{p(p-2)}{3}$ faces is the most faces that can be generated. □

Just as before, we now translate the bound on the number of faces into a bound on the genus of Cayley map embeddings of complete graphs with prime $p = 3k+2$ vertices.

Theorem 10.4. *For a complete graph K_p with prime p, such that $p = 3k+2$ for some integer k, a best possible Cayley map $C_M(Z_p, \rho)$ embeds K_p on a surface with genus $g \geq \lceil \frac{p(p-5)}{12} + 1 \rceil$.*

Proof. Let $C_M(Z_p, \rho)$ be a Cayley map for for the complete graph K_p with prime $p = 3k+2$ for some integer k. By definition, K_p has p vertices and $\frac{p(p-1)}{2}$ edges. Hence, for a Cayley map $C_M(Z_p, \rho)$ that produces F faces, the Euler characteristic is $\chi = p - \frac{p(p-1)}{2} + F$. Since $C_M(Z_p, \rho)$ for $p = 3k+2$ can generate at most $\frac{p(p-2)}{3}$ faces by Theorem 10.3, we have $\chi \leq p - \frac{p(p-1)}{2} + \frac{p(p-2)}{3}$. Using the genus formula and transitivity, we get $p - \frac{p(p-1)}{2} + \frac{p(p-2)}{3} \geq 2 - 2g$ for genus g. Solving for g, we get $g \geq 1 - \frac{p}{2} + \frac{p(p-1)}{4} - \frac{p(p-2)}{6} = \frac{p^2-5p+12}{12} = \frac{p(p-5)}{12} + 1$. Since g must be an integer, $g \geq \lceil \frac{p(p-5)}{12} + 1 \rceil$. □

11. Appendix A

Table 2 compares the optimal genus with the genus of a best possible Cayley map embedding for each complete graph discussed in this paper.

12. Appendix B

The following lemmas have not been used in this paper, but may prove useful in further research on best possible Cayley map embeddings of complete graphs.

Lemma 12.1. *Suppose H is an abelian group. For any Cayley map $C_M(H, \rho)$, if λ_1 is a k-cycle in λ that generates k-gons, no $k-1$ elements in λ_1 can add to 0.*

Proof. Suppose λ_1 is a k-cycle in λ that generates k-gons and $k-1$ elements in λ_1 add to 0. Let $x_1, x_2, \ldots, x_k \in \lambda_1$ and $x_1 + x_2 + \cdots + x_{k-1} = 0$. For λ_1 to generate k-gons, we know $x_1 + x_2 +$

Table 2. Genus of Cayley Map Embeddings of Complete Graphs

Complete Graph K_n	Best Cayley Map Genus g	Optimal Genus $\gamma(K_n)$	$g - \gamma(K_n)$
K_4	0	0	0
K_5	1	1	0
K_6	1	1	0
K_7	1	1	0
K_{11}	10	5	5
K_{13}	13	8	5
K_{17}	18	16	2
K_{19}	20	20	0
K_{31}	63	63	0
K_{43}	130	130	0
K_{55}	221	221	0
K_{67}	336	336	0
K_{79}	475	475	0
K_{91}	638	638	0
K_{103}	825	825	0
K_{115}	1036	1036	0

$\cdots + x_{k-1} + x_k = 0$. By substitution, we get $x_k = 0$, which is not possible by the nature of Cayley maps. Hence, no $k-1$ elements in λ_1 can add to 0. □

Lemma 12.2. *Suppose H is an abelian group, $X = \{x_1, x_2, \ldots, x_n\}$ is a closed subset of H, and ρ is a cyclic permutation of X. For the Cayley Map $C_M(H, \rho)$, if $\lambda_1 = (x_1, x_2, \ldots, x_k)$ for some even integer $k < \frac{n}{2}$, then $\lambda_2 \neq (-x_1, -x_2, \ldots, -x_k)$.*

Proof. We prove the contrapositive. Let $\lambda_1 = (x_1, x_2, \ldots, x_k)$ and $\lambda_2 = (-x_1, -x_2, \ldots, -x_k)$ for some even $k < \frac{n}{2}$. We see that $\rho(x_1) = -x_2$, $\rho(-x_2) = x_3, \ldots, \rho(-x_k) = x_1$ and $\rho(-x_1) = x_2$, $\rho(x_2) = -x_3, \ldots, \rho(x_k) = -x_1$ resulting in $\rho = (x_1, -x_2, x_3, \ldots, -x_k)(-x_1, x_2, -x_3, \ldots, x_k)$, which contradicts ρ being a cyclic permutation. Therefore, if $\lambda_1 = (x_1, x_2, \ldots, x_k)$ for some even integer $k < \frac{n}{2}$, then $\lambda_2 \neq (-x_1, -x_2, \ldots, -x_k)$. □

Lemma 12.3. *For any Cayley Map $C_M(Z_n, \rho)$ where n is odd, if λ has at least two factors λ_i and some $|\lambda_j| = 1$, then there must exist a $\lambda_k \neq \lambda_j$ such that $mult(\lambda_k) \neq 1$.*

Proof. Let λ have $k \geq 2$ factors and λ_1 have only one element, x_1, such that $|\lambda_1| = 1$. Since n is odd and $|\lambda| = n - 1$, every element of λ has a unique inverse element such that $Sum(\lambda) = 0$. Suppose $mult(\lambda_i) = 1$ for all $i \in \{2, \ldots, k\}$, meaning $Sum(\lambda_i) = 0$ for all i. Hence, $\Sigma_{i=2}^{k} Sum(\lambda_i) = 0$. However, $Sum(\lambda) = \Sigma_{i=1}^{k} Sum(\lambda_i) = Sum(\lambda_1) + \Sigma_{i=2}^{k} Sum(\lambda_i) = 0$. By substitution, $Sum(\lambda_1) =$

$x_1 = 0$. However, by definition, $x_1 \neq 0$, a contradiction. Therefore, $mult(\lambda_i) \neq 1$ for some $i \in \{2, \ldots, k\}$. □

Lemma 12.4. *For a Cayley map $C_M(Z_n, \rho)$ with odd n, if $\lambda_1 = (x)$ and $\lambda_2 = (y)$ are the only two non-multiplicity 1 factors of λ, then $\rho = (x, y)$ is a 2-cycle.*

Proof. Since n is odd, $Sum(\lambda) = 0$. Since $\lambda_1 = (x)$ and $\lambda_2 = (y)$ are the only non-multiplicity 1 factors of λ, $Sum(\lambda) - \lambda_1 - \lambda_2 = Sum(\lambda) - x - y = 0$. By substitution, $-x - y = 0$ so $y = -x$. Thus, $\lambda_1 = (x)$ and $\lambda_2 = (-x)$, which results in $\rho = (x, -x)$ such that ρ is a 2-cycle. □

13. Appendix C

Below is the Python code used to generate permutations λ and ρ of optimal Cayley map embeddings for complete graphs of the form K_{12k+7} for $k > 0$. The code can more generally be used for complete graphs K_n where $n = 1 \mod 3$ and $n > 7$. Note that this code does not generate all possible lambdas of optimal Cayley map embeddings, only those with 3-cycles of the form (i, j, k) such that $i < j < k$. It is also worth noting that as the value of n increases, the time it takes to run the program quickly increase past the reasonable capacity of a standard computer.

```
# Looks for optimal Cayley map embeddings of complete
# graphs Kn for n = 12k+7 for nonnegative integers k
# Cayley map is optimal if it generates only 3-gons
# So lambda will have numTriples order one 3-cycles

# Number of vertices n will change depending on the graph
n = 43
numTriples = (n-1)/3

# Returns True if k is in a triple in aList
def searchList(aList, k):
    if len(aList) == 0:
        return False
```

```
    return (k in aList[0]) or searchList(aList[1:], k)

# Finds rho using rho(x)=lambda(-x) for potential lambda
# If rho is not cyclic, only one cycle of rho is found
def createRho(potentialLambda):

    # Start rho with element x = 1
    rho = []
    x = 1
    rho.append(x)

    # A valid rho will be a cycle of n-1 elements
    for a in range(n-1):
        # Search potentialLambda for the triple containing element -x
        # The element after -x in the triple follows x in rho
        for i in range(len(potentialLambda)):
            if (n - x) in potentialLambda[i]:
                z = (potentialLambda[i].index(n - x) + 1) % 3
                x = potentialLambda[i][z]
                # Return rho when it is about to cycle back to 1
                if x == 1 and len(rho) != 1:
                    return(rho)
                rho.append(x)
    return(rho)

# Returns True if rho is indeed cyclic with n-1 elements
def isCayleyMap(rho1):
    return(len(rho1) == n-1)
```

```python
# Iteratively generates potential lambdas of order one triples
def genLambda(potentialLambda):
    x = 1
    y = 2

    # Restrict triples to the form (i,j,k) where i<j<k<n
    # Note: this means not all lambdas will be generated
    while potentialLambda[0][1] < potentialLambda[0][2]:

        # Potential lambda is "full" (has all elements < n)
        if len(potentialLambda) == numTriples:
            # Prints potential lambda and rho if rho is cyclic
            if isCayleyMap(createRho(potentialLambda)):
                print(potentialLambda)
                print(createRho(potentialLambda))
            # Backtrack to find next potential lambda
            potentialLambda.pop()
            temp = potentialLambda.pop()
            x = temp[0]
            y = temp[1] + 1

        # Fill potentialLambda with order one triples
        for i in range(x,n):
            if not searchList(potentialLambda,i):
                for j in range(y,n):
                    if not searchList(potentialLambda,j)
                    and not searchList(potentialLambda,i) and j > i:
                        k = (n - i - j) % n
                        if k > j and not searchList(potentialLambda, k)
```

```
                    and not searchList(potentialLambda, i)
                    and not searchList(potentialLambda, j):
                        potentialLambda.append([i,j,k])

            # Backtrack if i could not be added to a triple
            if not searchList(potentialLambda, i):
                temp = potentialLambda.pop()
                x = temp[0]
                y = temp[1] + 1
                # potentialLambda cannot be empty
                if len(potentialLambda) == 0:
                    z = (n - x - y) % n
                    potentialLambda.append([x,y,z])
                    x += 1
                    y = x+1
                break
    return

# Iteratively, the first triple starts as firstTriple
# Alternatively, you can pass any potential lambda to genLambda and
# it will continue iterating from there on
firstTriple = [[1,2,n-3]]
lambdas = genLambda(firstTriple)
```

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