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Coupling of Transverse and Longitudinal Waves in Piano Strings

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Coupling of transverse and longitudinal waves in piano strings

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The existence of longitudinal waves in vibrating piano strings has been previously established, as has their importance in producing the characteristic sound of the piano. Modeling of the coupling between the transverse and longitudinal motion of strings indicates that the amplitude of the longitudinal waves are quadratically related to the transverse displacement of the string, however, experimental verification of this relationship is lacking. In the work reported here this relationship is tested by driving the transverse motion of a piano string at only two frequencies, which simplifies the task of unambiguously identifying the constituent signals. The results indicate that the generally accepted relationship between the transverse motion and the longitudinal motion is valid. It is further shown that this dependence on transverse displacement is a good approximation when a string is excited by the impact of the hammer during normal play. © *2015 Acoustical Society of America*.
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I. INTRODUCTION

Modeling the sound of the piano requires a quantitative understanding of each of the components involved, as well as the interactions between them. Recently, Chabassier *et al.* have proposed an extensive model that includes many of the interactions between the constituent parts of the piano.[1] Of particular importance to this work, as well as other efforts to model the piano, is the manner in which the string motion is simulated. Arguably, the string motion is the most important component in creating the piano sound, therefore an understanding of the motion of a struck string is critical to producing an accurate model.

The motion of strings clamped at both ends is well understood, however, this situation is only an approximation to the motion of the strings in a fully assembled piano because the finite impedance of a piano bridge means that one end of the string is not completely immobile. Other complications include the fact that the string extends beyond the bridge pin, most strings occur in pairs or triplets, and some are wrapped in copper wire to increase the linear density.[2–5] In addition to the recent focus on these complications, research within the past two decades has shown that piano strings vibrate in the longitudinal direction, which produces significant audible effects.[3,6] The origin of these longitudinal vibrations and their relationship to the transverse motion are the subject of the work reported here.

The presence of transverse waves in vibrating piano strings has been both theoretically investigated and experimentally verified, and a complete quantitative understanding is near. However, questions regarding the origins of longitudinal waves in piano strings remain unanswered. Recent studies indicate that longitudinal waves significantly contribute to the piano sound,[6] but the excitation mechanisms that initiate these waves are still not completely understood.[7] Theoretical work on the longitudinal vibrations in strings dates back to the middle of the last century,[8] and there has been significant progress within the last 20 years.[9–13] This body of work has resulted in a model describing the induced longitudinal motion that is generally accepted by the scientific community, but to our knowledge there has been no experimental evidence to support it.

It appears that the presence of audible longitudinal vibrations in the sound of a piano was first reported by Knoblaugh in 1944.[14] Knoblaugh referred to the resulting sound as a clang tone and posited that the motion of the hammer tangent to the string induces the longitudinal vibrations that produce it. Over 50 years later, unaware of Knoblaugh's work, Conklin investigated longitudinal vibrations in piano strings. He referred to the sounds attributable to these vibrations as *phantom partials* and suggested that these inharmonic partials were due to longitudinal waves produced by nonlinear coupling with the transverse motion.[15] Giordano and Korty experimentally investigated these longitudinal vibrations by measuring the motion of the bridge and soundboard as the hammer strikes a piano string.[7] They concluded that the magnitude of the longitudinal motion is nonlinearly related to the transverse displacement induced by the hammer, but unfortunately they could not conclusively determine the relationship between them.

The theoretical basis for the nonlinear coupling of transverse and longitudinal motion in strings was first discussed by Morse and Ingard in 1968.[8] Based on this work, Bank and Sujbert noted that the amplitude of the longitudinal vibrations should be quadratically related to the transverse displacement of the string, and that although the measurements of Ref. 7 did not explicitly identify the power of the nonlinearity, their work did confirm a nonlinear relationship between the transverse and longitudinal excitation.[16] More recent work has added a more rigorous theoretical basis,[11–13] but experimental evidence has been lacking.

In the work reported here we confirm that longitudinal standing waves can be produced by the nonlinear coupling

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of transverse standing waves in real piano strings. Furthermore, we show that the acoustic power of the sound produced by these longitudinal vibrations in a piano is linearly proportional to the power in each of the two transverse waves that produce them. Therefore, when the amplitudes of two transverse waves are linearly related, as they may be when they are produced by the piano hammer striking the string, the amplitude of the longitudinal wave is proportional to the square of the transverse string displacement.

In what follows we first briefly review the theoretical basis for the coupling between transverse and longitudinal waves in strings. We then present the results of experiments indicating that the model can accurately predict the motion of a piano string *in situ*. Specifically, by driving a piano string such that only two transverse waves exist on the string, we show that the longitudinal vibrations occur at the predicted frequencies and are linearly proportional to the amplitude of each of the transverse waves. We then provide evidence that this is an excellent approximation for predicting the response of a piano string when struck by a hammer during normal play.

II. THEORY

Two types of longitudinal waves have previously been identified in piano strings. The first are referred to as free-response longitudinal waves, and they occur at frequencies associated with the longitudinal resonance frequencies of the string. The frequencies of these resonances are determined by the length of the string and the speed of sound in the material. The second type is termed a forced-response longitudinal wave; these waves are induced by the nonlinear mixing of transverse waves in the string and can occur at frequencies other than those associated with the longitudinal resonances. The theory describing the nonlinear motion of the string was originally addressed by Morse and Ingard[8] and recently expanded upon by several authors, including Bank and Sujbert,[16] Bilbao,[11] and Chabassier and Joly.[13] Alternative theories also exist, see, for example, Refs. 5 and 9, however, they all result in the same predictions concerning the dependence of longitudinal waves on the transverse displacement. Here we do not reproduce the theory in detail, but only seek to provide enough background to understand the experimental evidence presented in Sec. III.

An element of a piano string at equilibrium with length dx and a corresponding element of a stretched piano string with length ds are shown in Fig. 1, where y and ξ are the transverse and longitudinal displacements of the string, respectively. It has been shown that by expanding both y and ξ as a series of polynomials and truncating at third order, the added force per unit length on the element in the longitudinal direction caused by the transverse displacement is given by[11,13,16]

$$F_x = ES\frac{\partial^2 \xi}{\partial x^2} + \frac{ES - T_0}{2}\left[\frac{\partial(\partial y/\partial x)^2}{\partial x}\right], \tag{1}$$

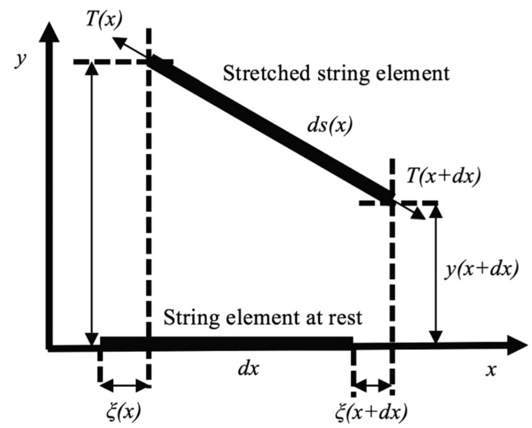

FIG. 1. Diagram of a stretched string element and the element at rest.

where E is Young's modulus, S is the cross sectional area of the string, and T_0 is the tension on the string when there is no displacement. A complete derivation of Eq. (1) can be found in Appendix B of Ref. 13.

The first term on the right hand side of Eq. (1) represents the component of the force in the direction parallel to the string caused by elongation of the string. This term is related to the initiation of free-response longitudinal vibrations. The second term in Eq. (1) represents the longitudinal force directly related to the transverse displacement, i.e., the forced-response.

Although it is not a computationally efficient technique, to better understand the results of the experiments described in Sec. III it is convenient to write the transverse motion of the string as a linear combination of the normal modes of the string, which constitute an orthonormal complete set.[10,12,16] To understand how the transverse displacement is related to the force in the longitudinal direction, we assume the normal modes of the string can be approximated by

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin(\omega_n t)\sin(k_n x), \tag{2}$$

where A_n is the amplitude of the nth transverse mode, k_n is the wave number, ω_n is the angular frequency, and t represents time. For a string pinned at both ends $k_n = n\pi/L$, where L is the length of the string. This is only an approximation for a piano string because there is non-infinite impendence at the bridge, however, for the purpose of this analysis the approximation is sufficient.

To better understand the system we limit the number of transverse modes in the piano string to 2. We refer to the modal angular frequencies as ω_n and ω_m, where n and m are the transverse mode numbers. In this case the transverse motion of the piano string can be described by

$$y(x,t) = A_n \sin(\omega_n t)\sin(k_n x) + A_m \sin(\omega_m t)\sin(k_m x). \tag{3}$$

Substituting Eq. (3) into the second term on the right hand side of Eq. (1) yields an equation for the force per unit length on the element dx in the longitudinal direction that is attributable to the transverse displacement,

$$F_{x,y} = \frac{-\pi^3(ES-T_0)}{L^3}\Bigg[A_m^2 m^3\cos(k_m x)\sin(k_m x)\sin^2(\omega_m t) + \frac{A_m A_n mn}{2}\{[\cos([\omega_n-\omega_m]t)+\cos([\omega_n+\omega_m]t)] \times [m\cos(k_n x)\sin(k_m x)+n\cos(k_m x)\sin(k_n x)]\} + A_n^2 n^3\cos(k_n x)\sin(k_n x)\sin^2(\omega_n t)\Bigg]. \quad (4)$$

The frequencies of the longitudinal waves induced by the transverse displacement are seen in Eq. (4) to be $2\omega_m$, $2\omega_n$, $\omega_n+\omega_m$, and $\omega_n-\omega_m$. The first two frequencies are harmonics of the driving frequencies and therefore are usually associated with both longitudinal and transverse motion. It is extremely difficult to experimentally determine the magnitude of the power in these frequencies due to purely longitudinal motion. Furthermore, it is reasonable to assume that in most cases the transverse motion of the string dominates the resulting motion of the bridge at these frequencies. Therefore, we focus on the second two frequencies.

These are the sum and difference of the transverse driving frequencies and do not necessarily overlap with any transverse vibrational frequencies. The analysis is simplified by isolating the two terms of interest in Eq. (4),

$$F_{x,y(+,-)} = \beta_{m,n}A_m A_n\{\cos([\omega_n+\omega_m]t) + \cos([\omega_n-\omega_m]t)\}, \quad (5)$$

where all of the constants except for the mode amplitudes are absorbed into $\beta_{m,n}$. An examination of Eq. (5) indicates that when the modal amplitudes are linearly related, there is a quadratic relationship between the amplitude of the induced longitudinal waves at the sum and difference frequencies and the amplitude of the motion associated with the transverse displacement of the string. An experimental investigation of this result is described in Sec. III.

III. EXPERIMENTS AND ANALYSIS

Previous investigations of the forced-response longitudinal waves in piano strings used the depression of the piano key to generate string motion.[7,17] When the string motion is excited in this manner it is the impulse of the felt hammer that initiates the motion, which results in a complex motion created by the many transverse modes that are simultaneously excited. Not only does impulsive initiation of the transverse motion produce numerous over-tones, which are subsequently transferred to the soundboard, but many of these overtones occur near the frequencies of the forced-response longitudinal waves. This complexity makes the identification of the frequency components associated with forced-response longitudinal waves difficult. The stiffness of the string can detune the overtones associated with the transverse motion so that the two motions occur at different frequencies, but this detuning is not significant for the lower overtones.

To experimentally investigate the process described by Eq. (5) it is necessary to ensure that no transverse overtones occur within the bandwidth of the forced-response waves. Since this is an unlikely scenario when the string is impulsively excited, it is prudent to replace the hammer excitation with steady-state harmonic excitation. By ensuring that the transverse motion of the string is driven at only two frequencies it is possible to isolate the signature of the forced-response longitudinal waves, which occur far from the driving frequencies and their harmonics.

To induce transverse oscillations in a string at only two frequencies an electromagnetic shaker was securely fastened to each end of the B_0^b string of an upright piano as shown in Fig. 2. The string was a wrapped, single string with a measured fundamental frequency of 27.4 ± 0.1 Hz. The speaking length of the string was 129.6 ± 0.2 cm, with wire wrapping that covered 125.5 ± 0.2 cm. Underneath the outer wrapping was an inner wrapping that covered 122.6 ± 0.2 cm of the wire. The core wire had a diameter of 1.400 ± 0.005 mm and a linear density of 0.0126 ± 0.0001 g/mm. The outer wrapping extended over the length of the inner wrapping as well as an additional 1.45 ± 0.01 cm on either end and had linear density of 0.116 ± 0.001 g/mm. The inner wrapping had a linear density of 0.0199 ± 0.0003 g/mm. One shaker drove the string at the frequency of the 16th overtone of the fundamental frequency of the transverse motion while the other drove it at the 19th overtone. The driving voltage of one shaker was held constant while the other was linearly ramped to the maximum allowable value. A type-1 condenser microphone was placed near the soundboard to record the audio signals, from which a power spectrum was calculated. A typical power spectrum is shown in Fig. 3. All experiments were performed in a hemi-anechoic chamber and sufficient time was allowed between experiments for the string to come to rest.

As expected, the four frequencies described in Eq. (4), $2\omega_m$, $2\omega_n$, and $\omega_n \pm \omega_m$, were observed to have significant power as shown in Fig. 3. Note that both the sum and difference frequencies are isolated from the driving frequencies and their overtones. In what follows we analyze the response of the string at a frequency equal to the sum of the two transverse driving frequencies, but a similar analysis of the difference frequency is also possible. Unfortunately, the presence of small harmonic distortion in the drivers induced transverse string motion at the frequencies $2\omega_m$ and $2\omega_n$. Since the efficiency of the transfer of string motion to bridge

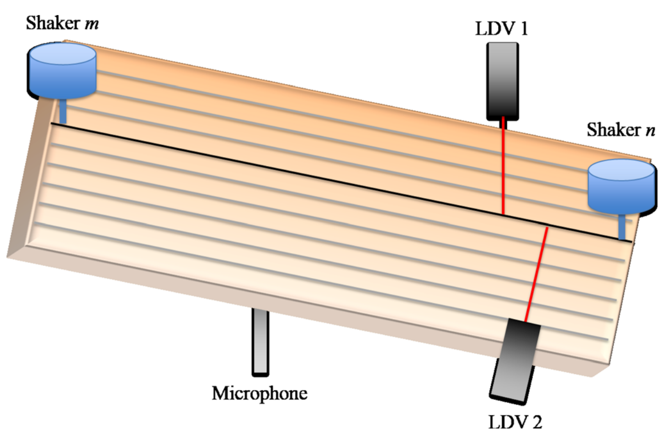

FIG. 2. (Color online) Schematic of the experimental arrangement. Only a portion of the piano soundboard is shown.

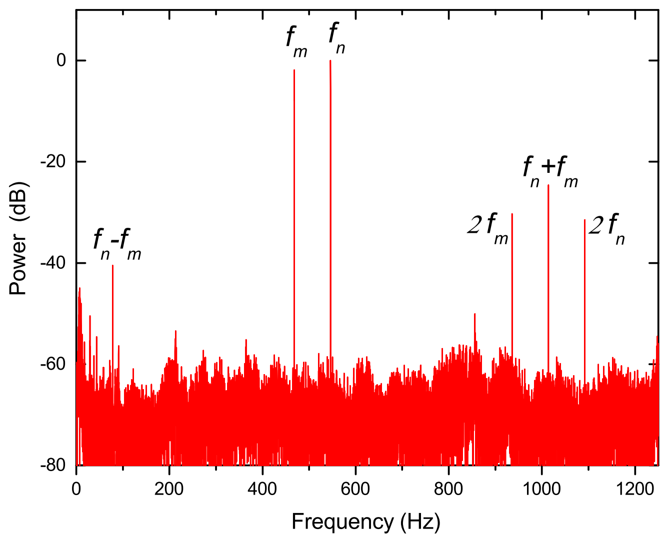

FIG. 3. (Color online) Power spectrum of the sound produced by a piano string driven transversely at two frequencies. The driving frequencies are approximately 468 Hz and 546 Hz.

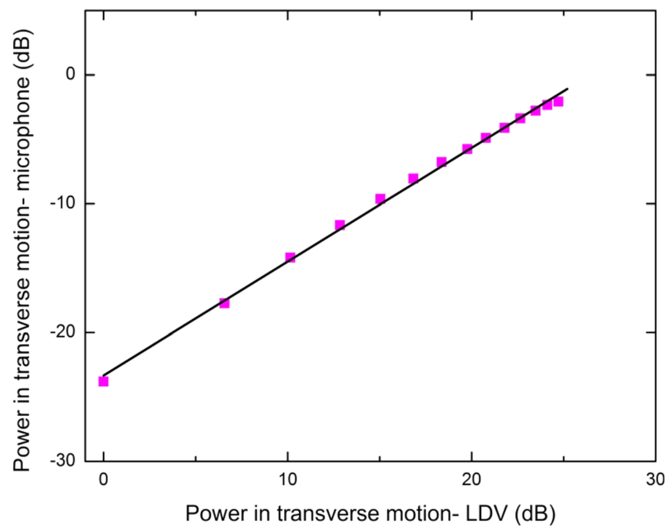

FIG. 4. (Color online) Power in a transverse frequency component recorded with the microphone plotted against the power in the same frequency component derived from LDV measurements of the string motion. The line represents a linear regression of the measurements.

motion is different for longitudinal and transverse waves, and the differences are not completely understood, it is not possible to determine the magnitude of the power at these frequencies that is attributable solely to longitudinal motion of the string.

The power in the sum and difference frequencies of the 16th and 19th overtones were reduced to the level of the noise when the string was driven at only one transverse frequency, confirming that both transverse waves are required to generate waves at these frequencies. Additionally, two laser Doppler vibrometers (LDVs) were directed perpendicular to the string along orthogonal axes to measure the transverse string motion. The power in the sum and difference frequencies did not rise above the level of the noise in the power spectra calculated from either of the LDV measurements. Therefore, the power in these frequency components can be unambiguously attributed to longitudinal waves in the string.

To simplify the analysis, the power in the transverse motion determined from measurements by the LDV at both driving frequencies was compared to the power recorded by the microphone at the same frequencies. Results of this comparison are shown in Fig. 4, which indicates that a linear relationship exists between the two measurements. Therefore, the LDV measurements are redundant and only audio measurements are required to analyze the system. This not only simplifies the current experimental arrangement, but it indicates that measurements of the sound produced by the piano are sufficient for future investigations of this type.

To demonstrate that Eq. (5) is valid and that the longitudinal vibrations are indeed linearly proportional to the amplitudes of the transverse driving waves, the power in each of the individual driving frequencies and the power in the sum frequency is graphed as a function of the power in the driving frequency of the ramped excitation in Fig. 5. Since the amplitude of one shaker was held constant and the other was linearly ramped, the linear relationship between the acoustic power of the forced-response longitudinal wave and the power in the transverse motion implies that Eq. (5) accurately describes the experimental situation. Furthermore, it indicates that there is a quadratic response when both driving amplitudes are increased simultaneously. When the amplitudes of both shakers were linearly ramped simultaneously, the quadratic relationship between the power in the longitudinal frequency components and the power in the transverse displacement, shown in Fig. 6, is clear.

The measurements shown in Figs. 5 and 6 imply that Eq. (5) accurately describes the system with steady-state, low-amplitude excitation. However, one may question whether Eq. (5) is applicable to a fully assembled piano under normal playing conditions. That is, does this relationship hold for the case of impulse excitation by the piano hammer?

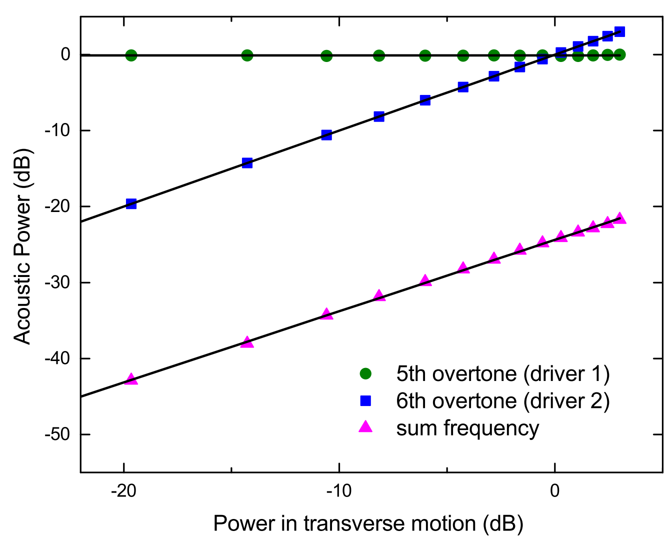

FIG. 5. (Color online) The power in the two driving frequencies and the sum frequency when the amplitude of one driver is held constant and the other is linearly ramped. The power is plotted as a function of the power in the frequency of the ramped oscillator. All measurements have been normalized to the power in the frequency of oscillation that was held constant. The lines represent linear fits.

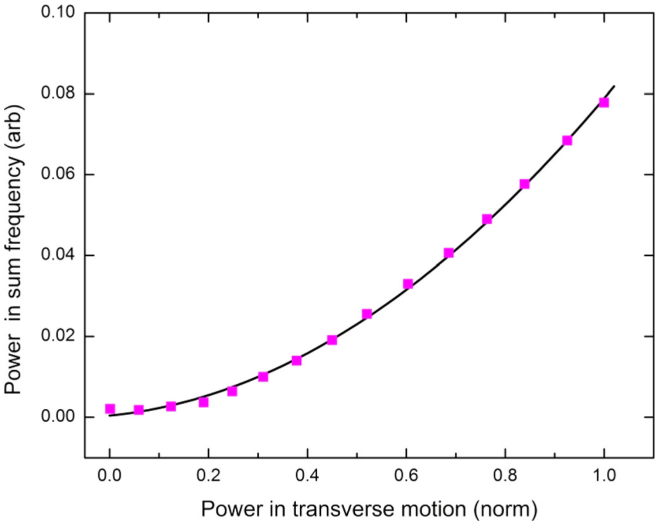

FIG. 6. (Color online) The power in the longitudinal frequency component plotted against the power attributable to the transverse motion. The line represents a quadratic fit to the data.

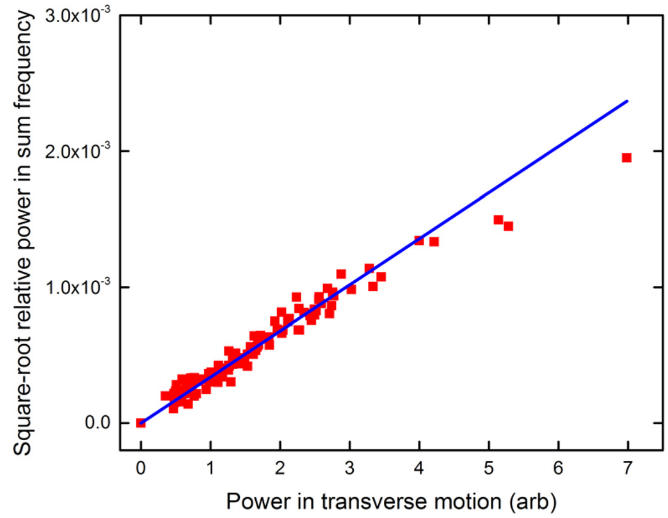

FIG. 8. (Color online) The square-root of the power in the frequency component associated with a longitudinal wave plotted as a function of the total power in the fundamental and first 14 overtones of the transverse motion when the motion is initiated by depressing the piano key. The total power in the transverse motion is proportional to the transverse displacement of the string while the amplitude of the longitudinal motion is proportional to the power in the sum frequency. The line represents a linear regression of the measurements excluding the five points with the highest value.

To determine if Eq. (5) is valid under normal playing conditions, the A0 string of a Steinway grand piano was depressed multiple times with various amounts of force, causing the corresponding piano hammer to strike the string and produce transverse vibrations. A microphone recorded the audio signals after each strike, which were used to calculate a power spectrum for each instance. The frequency associated with a longitudinal wave was then identified at a frequency that did not correspond to a transverse overtone.

The two transverse frequencies ω_m and ω_n were identified as the 10th and 12th overtones of the transverse fundamental. The frequency of the longitudinal wave at the sum of these frequencies did not overlap with the 22nd overtone of the transverse motion, but occurred between the 21st and 22nd overtones due to the increased frequency of the overtones attributable to the effects of the string stiffness. As shown in Fig. 7 the amplitudes of the two overtones were linearly related as the hammer force was increased, which indicates that the amplitudes of the transverse waves increase equally with increased force on the key. Saturation effects are noticeable at high forces, however, the amount of force used to depress the key in these instances was excessive and one would expect that forces of this magnitude are seldom used during performance.

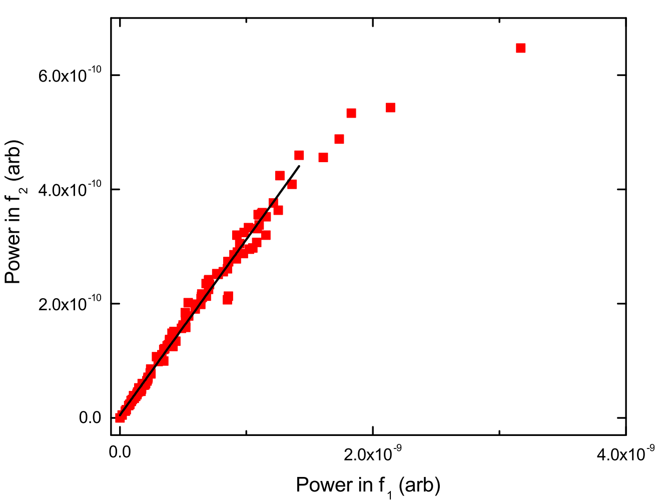

FIG. 7. (Color online) The power in the 12th overtone of the transverse fundamental as a function of the power in the 10th overtone for varying forces used to depress the piano key. The line represents a linear regression of the measurements excluding the five points with the highest value.

The results shown in Fig. 7 indicate that the amplitude of the longitudinal wave should increase quadratically with the amplitude of the transverse displacement of the string. Equivalently, the power in the sound produced by the longitudinal wave should increase quadratically with the sum of the power in the fundamental frequency of the transverse motion and all of its overtones. More simply, we expect a linear relationship between the square-root of the power in the frequency component generated by the longitudinal wave and the sum of the power in all of the frequency components directly attributable to the transverse motion. This relationship is shown in Fig. 8, where the power attributable to the transverse motion was determined by summing the power in the fundamental and first fourteen transverse overtones.

IV. CONCLUSIONS

The experimental results reported here support the validity of the model described in Sec. II, which has been used for several years without the support of empirical evidence. In a system where the transverse vibrations along a string are continuously driven, the power in the induced longitudinal motion is linearly related to each of the transverse driving amplitudes. Therefore, when the two transverse amplitudes increase simultaneously, the amplitude of the resulting longitudinal motion is quadratically related to the string displacement. The results shown in Fig. 8 indicate that this relationship also describes the motion of the string under normal playing conditions, however, it is evident that when the key is depressed with excessive force other processes can affect this relationship.

We note in closing that N. Giordano has pointed out in a personal communication that the analysis of Sec. II may apply equally well to the motion of the bridge as it does to the string motion. Determining the relative magnitudes of the coupling through the bridge and the coupling through the string will entail further experimentation.

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