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The dynamics and tuning of orchestral crotales

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An experimental and theoretical investigation of the acoustic and vibrational properties of orchestral crotales within the range C_6 to C_8 is reported. Interferograms of the acoustically important modes of vibration are presented and the frequencies are reported. It is shown that the acoustic spectra of crotales are not predicted by assuming that they are either thin circular plates or annular plates clamped at the center, despite the physical resemblance to these objects. Results from finite element analysis are presented that demonstrate how changing the size of the central mass affects the tuning of the instruments, and it is concluded that crotales are not currently designed to ensure optimal tuning. The possibility of using annular plates as crotales is also investigated and the physical parameters for such a set of instruments are presented. © 2004 Acoustical Society of America. [DOI: 10.1121/1.1788728]

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I. INTRODUCTION

In the field of musical acoustics percussion instruments are understood especially well. To our knowledge, however, there is a nearly complete absence of discussion of orchestral crotales in the literature. The single exception appears to be a short mention of their acoustic properties by Fletcher and Rossing.¹ While the term *crotale* can be associated with several different types of percussion instruments, commercially available orchestral crotales offer little diversity; they are small cymbals with a central mass, as illustrated in Fig. 1. These instruments are commonly found in orchestras around the world and are commercially produced in the United States by at least two large manufacturers of percussion instruments. Each crotale in a set is tuned to one note of the Western musical scale and the note is stamped onto it for identification. The sound is usually produced by striking the instrument with a mallet.

Crotales have a particularly pleasing sound, owing to the fact that the dominant partials are the second, fourth, and seventh harmonics of a nonexistent fundamental. The fortunate arrangement of these partials is clearly the result of the presence of the central mass, but to our knowledge there is no published discussion of the subject.

While the outer radii of the crotales become smaller as the pitch increases as one would expect, the radii of the central masses of the crotales do not change between C_6 and C_8 . The invariability of the radius of the center mass leads one to question whether the tuning of each crotale is optimal. Here we report on an investigation of the acoustic and vibrational properties of a set of crotales in the two octaves from C_6 to C_8 , identifying the vibrational modes and assessing their relative importance to the sound of the instruments. Evidence is presented demonstrating that the crotales are indeed not optimally tuned.

To assess the importance of the center mass on the normal modes of the crotale well-established thin plate theory is used to predict the normal mode frequencies. There is poor

agreement between predicted and empirical values, and we conclude that the presence of the center mass is responsible for this discrepancy.

The center mass of a crotale causes it to physically resemble an annular plate that is free to vibrate at the outer radius and clamped at the inner radius. Rigid mounting through the center hole reinforces this resemblance. We therefore develop a model of the crotales as annular plates. Again, well-established theory is used to compare theoretical predictions with experimental results, and they are again found to be in poor agreement, although the agreement is better than is found when comparing experimental results to thin plate theory.

We finally turn to finite element analysis to facilitate an understanding of the effects of the center mass on the tuning of crotales. Within the model the height and radius of the central mass of the crotales are varied. The effect of these changes on the tuning of the crotales is then analyzed.

We conclude by presenting an alternative design for the manufacture of crotales based on annular plate theory. We present physical parameters for annular plates such that they have similar acoustic properties to crotales. The validity of this design as an alternative to commercially available crotales is confirmed using finite element analysis.

II. EXPERIMENT

A. Identification of the acoustically important modes

The crotales used in this investigation were manufactured by Zildjian Co. They span the octaves from C_6 to C_8 and have diameters ranging from 132.8 to 76.5 ± 0.1 mm. All of the central masses are identical and have a diameter of 29.3 ± 0.1 mm and a thickness of 13.1 ± 0.1 mm. The thickness of the thin plate portion of the crotales is also uniform, measuring 4.7 ± 0.1 mm. In order to determine the acoustic properties of the crotales each one was mounted on a one-inch diameter vibration-damping post that was secured to a vibration-isolated optical table in an anechoic chamber. The crotale was struck with a cork mallet and the sound was

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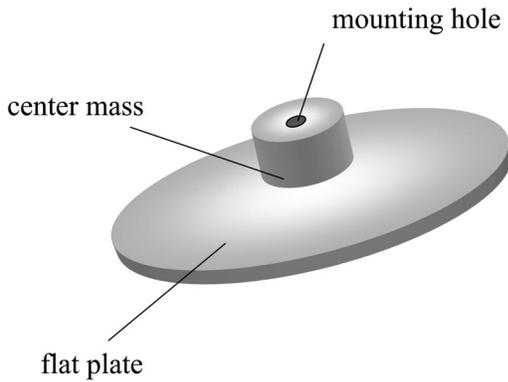


FIG. 1. Illustration of a crotales.

digitized. Two time series were recorded at a sampling rate of 40 kHz for each crotales in the set. The first series began at the strike time and had a duration of 0.25 s. A power spectrum of this time series was used to approximately identify the frequencies of many of the normal modes. The second time series was begun two seconds after the strike and had a duration of two seconds. Since this time series began after the transient modes had decayed to negligible relative power, the modes that are important in the steady-state sound of the crotales were evident.

Figure 2 is a typical example of the steady-state power spectrum of a struck crotales. Three modes are clearly visible, with most of the power being contained in the first two modes. As is commonly seen in other percussion instruments, the degenerate mode doublets are occasionally split due to slight asymmetries in the plate;² this is clearly evident in one of the modes shown in Fig. 2. Results from the other crotales within the set are similar, though in some cases the third mode is negligibly small and often none of the degenerate modes exhibit measurable splitting. We define an acoustically important mode as one that contains at least one percent of the total power, and using this definition there are at most three acoustically important modes for each crotales. In excess of 95% of the total power is contained within these three modes for all of the crotales, with no other single mode containing more than a small fraction of a percent of the total power.

Time-averaged electronic speckle pattern interferometry

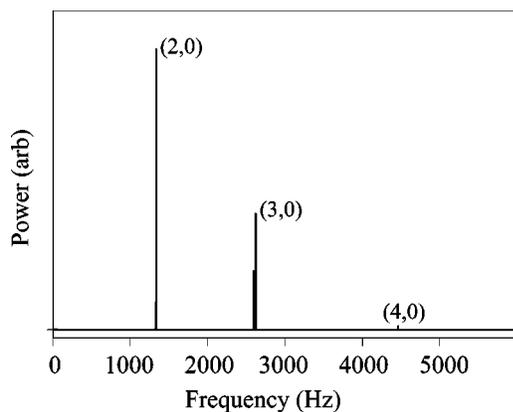


FIG. 2. Typical power spectrum of a crotales. The three acoustically important modes are clearly visible.

TABLE I. Diameters and frequencies of acoustically important modes of the two-octave set of crotales. Frequency uncertainties are ± 0.25 Hz.

Note	Diameter ± 0.1 mm	(2,0) mode	(3,0) mode	(4,0) mode
C_6	132.8	1055.5	2117.5	3667.5
$C^\#$	130.1	1117.0	2230.0	3849.5
D	125.9	1184.0	2348.0	4049.5
$D^\#$	123.8	1259.0	2483.0	4270.0
E	120.8	1333.5	2621.0	4502.5
F	117.8	1411.5	2754.5	4731.5
$F^\#$	114.6	1495.5	2879.0	4927.0
G	111.5	1585.5	3021.5	5166.5
$G^\#$	108.0	1682.0	3145.5	5361.0
A	106.6	1784.5	3357.5	5708.5
$A^\#$	104.6	1889.5	3548.5	6033.5
B	103.2	1668.5	3743.0	6340.0
C_7	101.6	2113.5	4043.5	6328.5
$C^\#$	98.0	2237.5	4214.5	6717.0
D	97.0	2361.0	4343.0	7082.5
$D^\#$	95.8	2521.5	4640.0	7554.5
E	91.6	2682.0	4777.5	8035.5
F	90.3	2832.5	5072.0	8492.0
$F^\#$	89.0	3004.0	5400.0	9011.5
G	86.2	3187.5	5609.0	9519.0
$G^\#$	84.4	3383.0	6106.0	10112.5
A	82.9	3584.5	6432.0	10792.0
$A^\#$	81.7	3801.0	6788.0	11224.0
B	79.5	4021.0	7144.0	11664.0
C_8	76.5	4241.0	7380.0	11992.0

was used to characterize the vibrational patterns of the modes of the crotales.³ To drive the vibrations, a speaker was placed in the anechoic chamber containing the crotales. The speaker was driven by a high-quality sine-wave generator. It was demonstrated in a similar experiment that the location and orientation of the speaker in the chamber does not affect the modal structure of the vibrations of the object under investigation.² However, in order to drive the vibrations with the maximum possible efficiency, the speaker was oriented perpendicularly to the face of the crotales. Using this method the three acoustically important modes were identified as the (2,0), (3,0), and (4,0) modes, where the integers represent the number of diametric and circular nodes, respectively. This procedure was repeated for each crotales in the set. The frequencies of these modes for each crotales are shown in Table I. Note that the ratios of the frequencies of the (3,0) to (2,0) modes are approximately 2:1 while the ratios of the frequencies of the (4,0) to (2,0) modes are approximately 7:2 for each crotales. Typical electronic speckle pattern interferograms are presented in Fig. 3. The center mass is not visible

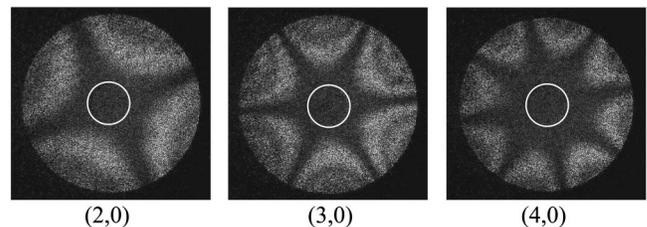


FIG. 3. Typical interferograms of the (2,0), (3,0), and (4,0) modes of a crotales. The light regions indicate places where the crotales is moving. Black regions indicate positions with little or no movement. The position of the center mass is indicated by a solid white line.

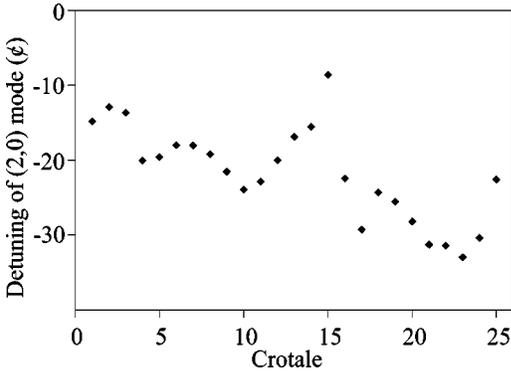


FIG. 4. Departure of the (2,0) mode from ideal tuning. If ideally tuned there would be no difference between the (2,0) mode and the frequency of the note according to the Western musical scale. The crotales correspond to numbers 1 through 25, respectively, with $C_6=1$ and $C_8=25$. Uncertainties are smaller than data points.

on the interferogram but is indicated by a solid white line in the figure.

B. Analysis of the tuning of crotales

Based on the data presented above, we define an ideally tuned crotales as one for which the (2,0) mode occurs at the frequency corresponding to the note name stamped on the crotales, and the frequency of the (3,0) mode is exactly one octave higher than that of the (2,0). Furthermore, the ratio of the frequencies of the (4,0) and (3,0) modes is 7:4, making a minor 7th. Using this definition, the tuning of each of the crotales was compared to the ideal. Figure 4 shows the detuning in cents of the (2,0) mode of each crotales from the frequency of its corresponding note on the usual chromatic scale.⁴ Generally, the (2,0) mode becomes less accurately tuned as the scale is ascended. Although the ability of a person to perceive a mistuned interval varies, a good musician can discriminate a 5 cents mistuning. Hall opines that it is reasonable to insist that an organ be tuned to within 2 or 3 cents of the target pitch;⁵ we see no reason for this standard not to be applied to crotales.

The ratios of the frequencies of the higher-order modes to the (2,0) mode show a similar trend as the scale is ascended. Figure 5 is a plot of the ratio of the frequencies of the (3,0) to (2,0) and (4,0) to (2,0) modes for each of the crotales, comparing each to the ideal 2:1 and 7:2 ratio. This demonstrates clearly that the crotales become increasingly detuned as the musical scale is ascended.

III. THEORY

A. Comparison to thin plate theory

Since crotales appear to be slightly modified thin plates fixed at the center it is reasonable to suspect that they can be accurately modeled using thin plate theory. Fortunately, thin plates have been studied for centuries and are well understood. Following the derivation presented by Leissa,⁶ the solution to the general equation of motion for a thin circular plate is

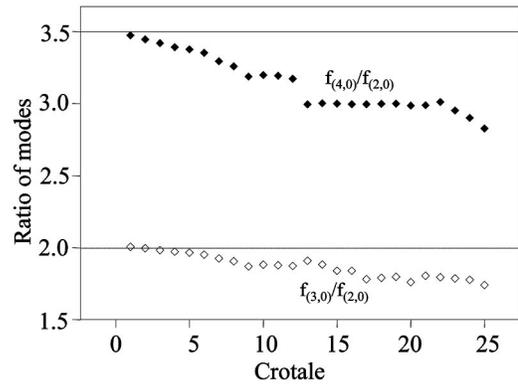


FIG. 5. Ratios of the frequencies of acoustically important modes. Open diamonds represent the ratios of the frequencies of the (3,0) to (2,0) modes. Closed diamonds represent the ratios of the frequencies of the (4,0) to (2,0) modes. The crotales correspond to numbers 1 through 25, respectively, with $C_6=1$ and $C_8=25$. Note the departure from ideal tuning of 2:1 and 7:2. Uncertainties are smaller than data points.

$$z = [A_1 J_n(kr) + A_2 I_n(kr) + A_3 Y_n(kr) + A_4 K_n(kr)] \cos(n\theta) \sin(\omega t), \quad (1)$$

where n is an integer, z is the deflection of a point from the equilibrium plane of the plate, θ and r are polar coordinates, A_i is a constant, J_n and I_n are Bessel functions of the first and second kinds, respectively, and Y_n and K_n are modified Bessel functions of the first and second kinds, respectively. The constant k in the argument of the Bessel functions is defined by

$$k^4 = \frac{12\rho\omega^2(1-\nu^2)}{Eh^2}. \quad (2)$$

Here, ν is Poisson's ratio, ρ is the volume mass density, E is Young's modulus, ω is the angular frequency, and h is the thickness of the plate. Terms with K_n and Y_n in Eq. (1) must be eliminated in this instance to prevent nonzero displacement at $r=0$, leaving only two unknown parameters in Eq. (1). Furthermore, for a plate fixed at the center, the $n=0$ terms are absent for the same reason. The boundary conditions at the edge of a plate of radius a that is free to vibrate are

$$M_r(a, \theta) = 0 \quad (3)$$

and

$$V_r(a, \theta) = 0, \quad (4)$$

where M_r is the bending moment, related to the displacement by

$$M_r(r, \theta) = -D \left[\frac{\partial^2 z}{\partial r^2} + \nu \left(\frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} \right) \right], \quad (5)$$

and V_r is the Kelvin–Kirchoff edge reaction, defined by

$$V_r(r, \theta) = -D \frac{\partial}{\partial r} (\nabla^2 z) + \frac{1}{r} \frac{\partial M_{r\theta}}{\partial \theta}. \quad (6)$$

Here, D is the flexural rigidity, defined as

TABLE II. The parameters used in all models to predict modal frequencies. The plate thickness (h) and density (ρ) were measured. Young's modulus (E) and Poisson's ratio (ν) are taken from Ref. 7.

Physical parameters of crotales	
E	10^{11} N/m ²
h	4.7 mm
ρ	8861 kg/m ³
ν	0.33

$$D = \frac{\rho \omega^2}{k^4 h}. \quad (7)$$

Upon applying these boundary conditions to Eq. (1), the eigenvalues determine the frequencies of the normal modes of the plates. We define a nondimensional frequency parameter as

$$\lambda = ka, \quad (8)$$

which can be used as a general solution for normal-mode frequencies, independent of the physical parameters of the plate.

In describing crotales as thin circular plates we assume that the physical parameters of every crotale are identical with the exception of the plate radius. Young's modulus and Poisson's ratio for yellow brass were taken from the literature;⁷ all other physical parameters were measured. The values of the parameters used are given in Table II. Table III lists values of λ and predicted frequencies of the acoustically important modes for the C_6 , C_7 , and C_8 crotales, as well as the error that results by comparing them to measured values. Note that it is not only the absolute frequencies that show poor agreement, but the ratios of the frequencies of the modes also do not agree with the experimental values. Clearly, this model is insufficient to predict the normal modes of crotales.

One possible explanation for this discrepancy is that the thickness of the plates in question violates the assumption of a thin plate. The thin plate theory outlined above applies only to plates for which the thickness is much less than the plate diameter. Although this appears to be a valid approximation given the physical parameters of the crotales, it is possible that it is not. In order to confirm the validity of modeling the crotales as thin circular plates the center masses of two of the crotales ($D_6^\#$ and F_6) were milled off to create a thin plate.

The frequencies of the normal modes were then experimentally determined and identified using the method described above and compared to predicted values for thin plates. The experimental values fall within 1.5% of the theoretical values for all but the (4,0) mode of the F_6 crotale, which deviates by approximately 2.4%. This supports the hypothesis that the crotales without the center mass may be modeled as thin plates. This also confirms that it is indeed the presence of the center mass that is responsible for the proper tuning of the crotales.

B. Comparison to annular plate theory

Since the crotales are physically clamped at the center when mounted for playing, and the interferograms shown in Fig. 3 indicate minimal movement of the central mass during play, one may suspect that crotales may be modeled as annular plates free to vibrate at the outer radius and clamped at the inner radius. To investigate this hypothesis further we model the crotale as an annular plate following the methods of Vogel and Skinner.⁸ Using an approach similar to the thin plate theory described above, we begin with Eq. (1); this time, however, the terms containing K_n and Y_n are allowed, since $r=0$ is not included as a boundary condition. Vogel and Skinner define the nondimensional frequency parameter for an annular plate to be

$$\lambda' = \omega \left(\frac{4\rho a^4}{Eh^2} \right)^{1/2}; \quad (9)$$

the relationship between λ and λ' is therefore

$$\lambda' = \frac{\lambda^2}{\sqrt{3(1-\nu^2)}}. \quad (10)$$

The boundary conditions for an annular plate clamped at the inner edge and free to vibrate at the outer edge are

$$\frac{\partial^2 z}{\partial r^2} + \nu \left(\frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} \right) = 0 \quad (11)$$

and

$$\frac{\partial}{\partial r} \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} \right) + \frac{1-\nu}{r^2} \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial z}{\partial r} - \frac{z}{r} \right) = 0, \quad (12)$$

TABLE III. Predicted frequencies of the acoustically important modes for three crotales using thin plate theory. The error when compared to the actual values is also indicated.

Diameter (± 0.1 mm) (Note)	Mode	λ	Predicted frequency (± 0.25 Hz)	Actual frequency (± 0.25 Hz)	% error	Ratio with (2,0) freq.
132.8 (C_6)	(2,0)	2.29	914.0	1055.5	13.40	1
	(3,0)	3.50	2135.1	2117.5	-0.83	2.33
	(4,0)	4.65	3768.7	3849.5	-2.75	4.11
101.6 (C_7)	(2,0)	2.29	1561.6	2113.5	26.11	1
	(3,0)	3.50	3647.8	4043.5	9.79	2.33
	(4,0)	4.65	6438.8	6328.5	1.74	4.11
76.5 (C_8)	(2,0)	2.29	2754.4	4241.0	35.05	1
	(3,0)	3.50	6434.2	7380.0	12.82	2.33
	(4,0)	4.65	11357.1	11992.0	5.29	4.11

TABLE IV. Predicted frequencies of the acoustically important modes for three crotales using annular plate theory. The error when compared to the actual values is also indicated.

Diameter (± 0.1 mm) (Note)	Mode	λ'	Predicted frequency (± 0.25 Hz)	Actual frequency (± 0.25 Hz)	% error	Ratio with (2,0) freq.
132.8 (C_6)	(2,0)	4.04	1150.7	1055.5	-9.02	1
	(3,0)	7.64	2178.4	2117.5	-2.87	1.89
	(4,0)	13.19	3758.9	3849.5	-2.49	3.27
101.6 (C_7)	(2,0)	4.68	2278.6	2113.5	-7.81	1
	(3,0)	7.92	3858.0	4043.5	4.59	1.69
	(4,0)	13.28	6466.2	6328.5	-2.18	2.84
76.5 (C_8)	(2,0)	6.03	5175.9	4241.0	-22.04	1
	(3,0)	8.78	7543.6	7380.0	-2.22	1.46
	(4,0)	13.71	11773.9	11992.0	1.82	2.27

for the free outer edge, and

$$z = 0 \quad (13)$$

and

$$\frac{\partial z}{\partial r} = 0, \quad (14)$$

for the clamped inner edge. When these conditions are applied to Eq. (1), the eigenvalues can be used to find the frequencies of the normal modes of the plates. The only physical parameters required to determine the eigenvalues are the ratio of the inner to outer radii and Poisson's ratio, assumed here to be 0.33 (Vogel and Skinner conclude that the eigenvalues are not sensitive to the value of Poisson's ratio).

Using this theory, we have predicted the frequencies of the (2,0), (3,0), and (4,0) modes for a set of free-clamped annular plates. The inner radii were chosen to be equal to the radii of center masses of the crotales and the outer radii were chosen to be equal to that of the indicated crotales. The values for λ' and predicted frequencies are shown in Table IV. Comparing these predictions to the actual frequencies of the crotales shows better agreement than was found for the model of a flat plate, indicating the superiority of the annular plate model. This is not surprising given the physical similarity between crotales and free-clamped annular plates; however, the poor agreement between the predicted ratios of the frequencies of the modes and the actual ratios of the frequencies demonstrates clearly that annular plate theory is also insufficient as a model for crotales.

Since these simple theories proved to be insufficient for predicting the behavior of crotales, we turned to finite element analysis to understand the effect of the center mass on the tuning.

C. Modeling the crotales using finite element analysis

A finite element model was developed using Solidworks, a commercially available software program. The model contained up to 127 620 nodes, with the actual number of nodes depending upon the size of the center mass relative to the plate. The software calculated the resonant frequencies of the normal modes as well as providing a visual confirmation of the mode shapes. This allowed us to compare predicted and observed mode shapes as well as the resonant frequencies.

The accuracy of the program was verified by modeling a thin flat circular plate. The predicted values matched those derived from Eq. (1) to within 0.5%. We then created a crotale model using the physical parameters of the C_6 crotale and compared the predicted resonant frequencies to those measured in the laboratory. The predicted frequency of the (2,0) mode using this model agrees to within 6% of the corresponding experimental value, and we believe that this error results from not knowing the exact values for Young's Modulus, and to a lesser extent Poisson's ratio. However, the ratios of the frequencies of the modes agree to within 0.7%.

The first physical parameter investigated within the model was the height of the center mass. A series of models of the C_6 and C_8 crotales were created with the height of the center mass ranging from a flat plate to twice the height of the actual center mass. The ratios of the acoustically important modes were then determined, and are plotted in Figs. 6 and 7. It is evident that increasing the height of the center mass has little effect on the ratios of the frequencies of the acoustically important modes of the C_6 crotale once it has reached 100% of the mass height as manufactured. However, the data in Fig. 7 indicate that any central mass detunes the smaller C_8 crotale, but with little extra effect after 100%.

The fact that increasing the height of the center mass beyond a certain point produces little or no change in the

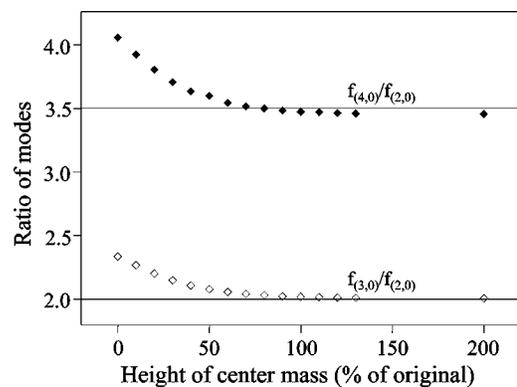


FIG. 6. Frequency ratios of the acoustically important modes of the C_6 crotale as a function of height of center mass according to the FEA model. Open diamonds represent the ratios of the frequencies of the (3,0) to (2,0) modes. Closed diamonds represent the ratios of the frequencies of the (4,0) to (2,0) modes. The ideal ratios of 2:1 and 7:2 are indicated by horizontal lines. After approximately 100%, the frequency ratios of the modes changes very little.

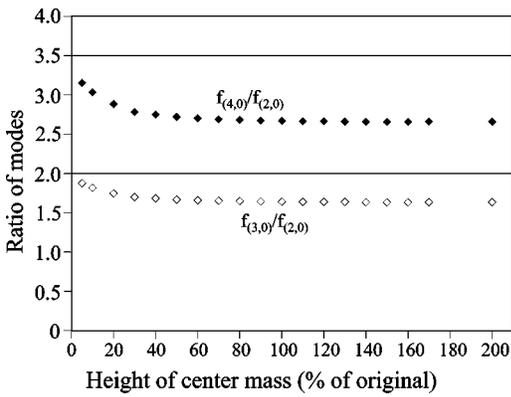


FIG. 7. Frequency ratios of the acoustically important modes of the C_8 crotale as a function of the height of the center mass according to the FEA model. Open diamonds represent the ratios of the frequencies of the (3,0) to (2,0) modes. Closed diamonds represent the ratios of the frequencies of the (4,0) to (2,0) modes. The ideal ratios of 2:1 and 7:2 are indicated by horizontal lines. Note that tuning worsens with the departure from a flat plate.

frequencies of vibration indicates that the center mass does indeed act as a clamping mechanism as was postulated above. However, clearly the boundary conditions are not equivalent to those of an annular plate clamped at the center.

The second investigation within the context of the finite element model entailed changing the radius of the center mass and determining the modal frequencies as described above. In this set of models the radius ranged from approximately 30% to 200% of the original center mass radius for the C_6 and C_8 crotales. The results of this investigation are plotted in Figs. 8 and 9. The radius of the center mass of crotales clearly has a significant effect on their tuning, as one would expect; however, these simulations indicate that the manufacturer could choose to make more ideally tuned crotales by choosing the height and radius for each crotale individually rather than a single size as they are currently manufactured. While the C_6 crotale is well-tuned as manufactured, both investigations of the C_8 crotale indicate that a flat plate of the same radius would be better tuned than the actual crotale.

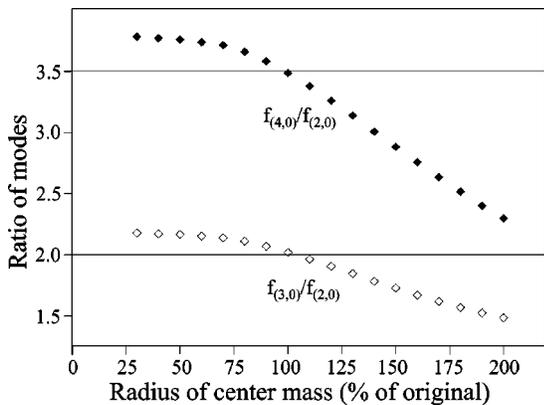


FIG. 8. Frequency ratios of the acoustically important modes of the C_6 crotale as a function of radius of center mass according to the FEA model. Open diamonds represent the ratios of the frequencies of the (3,0) to (2,0) modes. Closed diamonds represent the ratios of the frequencies of the (4,0) to (2,0) modes. The ideal ratios of 2:1 and 7:2 are indicated by horizontal lines.

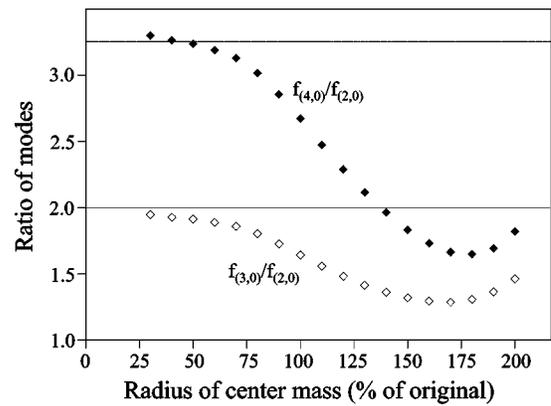


FIG. 9. Frequency ratios of the acoustically important modes of the C_8 crotale as a function of radius of center mass according to the FEA model. Open diamonds represent the ratios of the frequencies of the (3,0) to (2,0) modes. Closed diamonds represent the ratios of the frequencies of the (4,0) to (2,0) modes. The ideal ratios of 2:1 and 7:2 are indicated as horizontal lines. Again, tuning worsens with the departure from a flat plate.

D. An alternate solution for tuning crotales

While the annular plate model was shown to be insufficient for describing crotales, certain aspects of the theory deserve more careful consideration. The work of Vogel and Skinner implies that the ratio of the frequencies of the (3,0) to (2,0) mode becomes fixed when the ratio of the inner to outer radii is chosen. In most circumstances, the ratio of the frequencies of these modes corresponds to a unique value for the ratio of radii. This is true in the case of the (3,0) to (2,0) mode frequency ratio. Building on the work of Ref. 8, we have defined the parameters for a series of free-clamped annular plates that have similar acoustic properties to crotales. They have the added advantages of being more ideally tuned and containing less metal, presumably leading to lower production costs.

To meet the criteria for ideal tuning the (2,0) modes of the annular plates must occur at the frequencies corresponding to the desired note, a ratio of 2:1 must exist between the frequencies of the (3,0) and (2,0) modes, and a ratio of 7:2 must exist between the frequencies of the (4,0) and (2,0) modes. Since the (2,0) and (3,0) modes contain the most power, their nondimensional frequency parameters were chosen to optimize tuning. Using these criteria, the optimal relationship between the inner and outer radii was determined to be

$$\frac{b}{a} = 0.185, \quad (15)$$

where b is the inner radius. We note that this is a unique relationship and that it indicates a smaller ratio than exists for any of the crotales investigated if the inner radius is taken as the radius of the center mass. Once the ratio of radii is chosen, the ratio of nondimensional frequency parameters between any two modes is uniquely specified. The ratio of 0.185 corresponds to values of λ' for the acoustically important modes of

$$\lambda'_{2,0} = 3.79, \quad (16)$$

$$\lambda'_{3,0} = 7.57, \quad (17)$$

and

$$\lambda'_{4,0} = 13.18. \quad (18)$$

This ratio and set of nondimensional frequency parameters ensures that the ratio of the frequencies of the (3,0) to (2,0) modes is 2.00, and that the ratio of the (4,0) to (2,0) modes is 3.48. This is fortunate, since the latter results in an almost perfect minor seventh relationship and cannot be changed without affecting the ratio of the frequencies of the (3,0) to (2,0) modes. Note that since the nondimensional frequency parameter is related linearly to the frequency of its corresponding mode, the ratios of the frequencies of the modes are equal to the ratios of the nondimensional frequency parameters.

To finish the design all that remains is to choose the value of the outer radii of the plates such that the (2,0) modes occur at the correct frequencies. This can be accomplished by rearranging Eq. (9) to yield

$$a = \left(\frac{\lambda' h}{4 \pi f} \sqrt{\frac{E}{\rho}} \right)^{1/2}, \quad (19)$$

where f is the desired frequency of the (2,0) mode.

Using Eqs. (15) and (19), the radii of a set of free-clamped annular plates with tuning similar to that of an ideal crotale can be found. Values for the inner and outer radii of such a set of plates were determined in this manner, and finite element models were used to confirm the validity of these parameters. For both octaves, the ratios of the frequencies of the acoustically important modes agrees extremely well with the ideal. The ratio of the frequencies of the (3,0) to (2,0) modes agrees to within 1%, and the ratio of the frequencies of the (4,0) to (2,0) modes agrees to within 4% for the entire set. As one ascends the scale into the upper octave, however, the frequencies of the (2,0) modes are no longer accurately predicted by Eq. (19). This is apparently due to the thickness of the plates becoming large relative to the diameter. One may counter this effect by making the plates correspondingly thinner or decreasing the outer radius while keeping the same inner to outer radius ratio of 0.185.

IV. CONCLUSION

We have investigated the dynamics of orchestral crotales both theoretically and experimentally and have determined the important physical parameters in creating the sound of the instruments. We have shown that orchestral crotales typi-

cally have three acoustically important modes, which have been identified as the (2,0), (3,0), and (4,0) modes. However, in some cases only the (2,0) and (3,0) modes were found to be acoustically important. The frequencies of these modes were reported for each crotale in a two-octave set. The crotales become increasingly detuned as the Western musical scale is ascended, which may be explained by the invariance of the magnitude and radius of the central mass.

Empirical values of the frequencies of the normal modes of the crotales were compared to predicted values derived from a model of a thin plate clamped at the center and a model of an annular plate clamped at the center. Since both of these models were shown to be inadequate for describing the behavior of crotales, a finite element model of the crotales was used to investigate the importance of the central mass. It was found that decreasing the height of the center mass increases the ratios of the modal frequencies while increasing the height has little effect. Additionally, increasing or decreasing the radius of the center mass has a large effect on the tuning of a crotale. It has been shown that the physical parameters of the center mass have been chosen well for the lowest crotales, but that the highest crotale would be better tuned if the center mass were absent entirely.

Finally, a design for a more ideally tuned instrument was presented. This instrument consists of clamped annular plates with a ratio of inner to outer radii of 0.185.

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